

# Math 632 004 HW04

Shawn Zhong

TOTAL POINTS

**60 / 60**

QUESTION 1

## 1 Problem 1 10 / 10

✓ - 0 pts Correct

- 2 pts (a) One wrong detailed balance distribution
- 4 pts (a) Both detailed balance distributions are wrong
- 2 pts (b) Only asserted there are no additional invariant distributions for the first Markov chain without proof
- 2 pts (b) Did not find the other invariant distribution for the second Markov chain
- 2 pts (b) Did not give the right result of one of the Markov chain
- 4 pts (b) Did not give the right results of both Markov chains
- 10 pts No work submitted

QUESTION 2

## 2 Problem 2 10 / 10

✓ - 0 pts Correct

- 2 pts Some minor arithmetic or other computational errors
- 4 pts Setup the equations correctly but did not solve them
- 7 pts Made some effort, but there are substantial errors in the linear equations
- 10 pts No work submitted

QUESTION 3

## 3 Problem 3 10 / 10

✓ - 0 pts Correct

- 10 pts No work submitted

QUESTION 4

## 4 Problem 4: Durrett 1.56 10 / 10

✓ - 0 pts Correct

- 2 pts Some minor computational errors
- 4 pts The linear equations are set up, but there is no attempt to solve them
- 7 pts There are substantial errors in the way the linear equations are set up
- 10 pts No work submitted

QUESTION 5

## 5 Problem 5: Durrett 1.58 10 / 10

✓ - 0 pts Correct

- 2 pts (a) There are some mistakes in the transition probabilities
- 2 pts (a) The transient or recurrent states are not correctly identified
- 2 pts (b) The linear equations are not set up correctly
- 2 pts (b) The linear equations are not solved correctly
- 10 pts No work submitted.

QUESTION 6

## 6 Problem 6: Durrett 1.67 10 / 10

✓ - 0 pts Correct

- 2 pts (a) There are some mistakes in the transition probabilities
- 2 pts (a) The transient or recurrent states are not correctly identified
- 2 pts (b) The linear equations are not set up correctly
- 2 pts (b) The linear equations are not solved correctly
- 10 pts No work submitted

# HW4 - Problem & Solution

Friday, October 19, 2018 11:41 PM

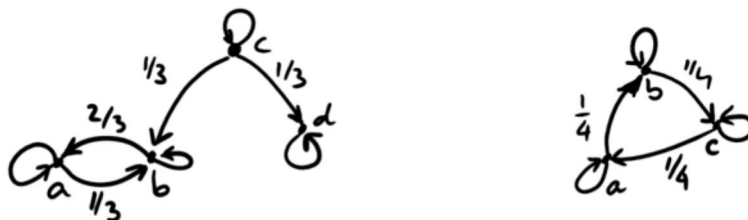
## Math 632 Lecture 4, Fall 2018, Homework 4 Due Thursday, October 25 by 10:05 AM.

There are six problems altogether in this homework set, three stated below and three problems from Durrett's textbook.

Please check the homework instructions on the course homepage. In particular, you should observe rules of academic integrity. You are encouraged to discuss the problems with fellow students, but everyone must hand in their own solution.

### Question 1

**Problem 1.** Consider these two Markov chains:



Address the following questions for each of the two Markov chains

- Find all invariant distributions that satisfy detailed balance.
  - Does the Markov chain have invariant distributions that do not satisfy detailed balance? Explain.
- For the Markov chain on the left
    - Let  $\pi$  be an invariant distribution
    - $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}^T \begin{bmatrix} 2/3 & 1/3 & 0 & 0 \\ 2/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}^T \Rightarrow \begin{cases} 2/3 x_1 + 2/3 x_2 = x_1 \\ 1/3 x_1 + 1/3 x_2 = x_2 \\ 1/3 x_3 = x_3 \\ 1/3 x_3 + x_4 = x_4 \end{cases} \Rightarrow \begin{cases} x_1 = 2x_2 \\ x_3 = 0 \\ x_4 = x_3 \end{cases}$$
    - Therefore the invariant distribution is  $\pi = \left[ \frac{2}{3}c \quad \frac{1}{3}c \quad 0 \quad 1-c \right]$  for some  $c \in [0,1]$
    - $$\begin{cases} \pi(1)p(1,2) = \pi(2)p(2,1) \\ \pi(1)p(1,3) = \pi(3)p(3,1) \\ \pi(1)p(1,4) = \pi(4)p(4,1) \\ \pi(2)p(2,3) = \pi(3)p(3,2) \\ \pi(2)p(2,4) = \pi(4)p(4,2) \\ \pi(3)p(3,4) = \pi(4)p(3,4) \end{cases} \Rightarrow \begin{cases} \frac{1}{3}\pi(1) = \frac{2}{3}\pi(2) \\ 0\pi(2) = \frac{1}{3}\pi(3) \end{cases} \Rightarrow \text{All invariant distribution satisfy DBE}$$
    - The invariant distribution satisfy detailed balance is  $\left[ \frac{2}{3}c \quad \frac{1}{3}c \quad 0 \quad 1-c \right]$  for  $c \in [0,1]$
  - For the Markov chain on the right
    - Let  $\pi$  be an invariant distribution

$$\circ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} 3/4 & 1/4 & 0 \\ 0 & 3/4 & 1/4 \\ 1/4 & 0 & 3/4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \Rightarrow \begin{cases} \frac{3}{4}x_1 + \frac{1}{4}x_3 = x_1 \\ \frac{1}{4}x_1 + \frac{3}{4}x_2 = x_2 \\ \frac{1}{4}x_2 + \frac{3}{4}x_3 = x_3 \end{cases} \Rightarrow x_1 = x_2 = x_3$$

$$\circ \text{Therefore the invariant distribution is } \pi = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\circ \begin{cases} \pi(1)p(1,2) = \pi(2)p(2,1) \\ \pi(1)p(1,3) = \pi(3)p(3,1) \\ \pi(2)p(2,3) = \pi(3)p(3,2) \end{cases} \Rightarrow \begin{cases} p(1,2) = p(2,1) \\ p(1,3) = p(3,1) \\ p(2,3) = p(3,2) \end{cases} \Rightarrow \pi \text{ does not satisfy DBE}$$

○ There is no invariant distribution that satisfy DBE

## Question 2

**Problem 2.** A Markov chain has four states  $\{A, B, C, D\}$ . Once the chain is in state  $D$  it stays there. From  $A$  or  $B$  the process jumps to one of the other three states with equal probability. From  $C$  the process jumps to  $A$  or  $B$  with equal probability.

Compute  $E_A[T_D]$ ,  $E_B[T_D]$ , and  $E_C[T_D]$ .

$$\bullet \mathcal{P} = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

• Let  $g(A) = E_A[T_D]$ ,  $g(B) = E_B[T_D]$ ,  $g(C) = E_C[T_D]$

$$\bullet \begin{cases} g(A) = 1 + \frac{1}{3}g(B) + \frac{1}{3}g(C) \\ g(B) = 1 + \frac{1}{3}g(A) + \frac{1}{3}g(C) \\ g(C) = 1 + \frac{1}{2}g(A) + \frac{1}{2}g(B) \end{cases} \Rightarrow \begin{cases} E_A[T_D] = g(A) = 4 \\ E_B[T_D] = g(B) = 4 \\ E_C[T_D] = g(C) = 5 \end{cases}$$

## Question 3

**Problem 3.** Consider the following graph: the vertices are the points

$$\mathcal{S} = \{(i, j) \mid i, j \in \{1, 2, 3, 4\}\}$$

Two vertices  $(i, j)$  and  $(i', j')$  are connected if either  $i = i'$ ,  $j = j' \pm 1$ , or  $j = j'$  and  $i = i' \pm 1$ .

The *core* of the graph is  $\mathcal{C} = \{(i, j) \mid i, j \in \{2, 3\}\}$ . The *edge* of the graph consists of all points that are not in the core.

Let  $X_n$  be the Markov process that describes a random walk on  $\mathcal{S}$ . Define  $T = \min\{n \geq 0 \mid X_n \in \mathcal{C}\}$ .

Explain how you can compute the probability  $P[X_T = B \mid X_0 = A]$  where  $A = (1, 1)$  and  $B = (3, 3)$ . You should provide enough detail so that someone with a computer or calculator could use your explanation to compute  $P[X_T = B \mid X_0 = A]$ .

Optional: use Matlab/Octave/Python to do the computation.

• Let  $V_B := \inf\{n \geq 0 \mid X_n = B\}$ ,  $V_{\mathcal{C} \setminus \{B\}} := \inf\{n \geq 0 \mid X_n \in \mathcal{C} \setminus \{B\}\}$ , and  $h(x) := \mathbb{P}_x(V_B < V_{\mathcal{C} \setminus \{B\}})$

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✓ - 0 pts Correct

- 2 pts (a) One wrong detailed balance distribution
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$$\circ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} 3/4 & 1/4 & 0 \\ 0 & 3/4 & 1/4 \\ 1/4 & 0 & 3/4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \Rightarrow \begin{cases} \frac{3}{4}x_1 + \frac{1}{4}x_3 = x_1 \\ \frac{1}{4}x_1 + \frac{3}{4}x_2 = x_2 \\ \frac{1}{4}x_2 + \frac{3}{4}x_3 = x_3 \end{cases} \Rightarrow x_1 = x_2 = x_3$$

$$\circ \text{Therefore the invariant distribution is } \pi = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

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• Let  $g(A) = E_A[T_D]$ ,  $g(B) = E_B[T_D]$ ,  $g(C) = E_C[T_D]$

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## 2 Problem 2 10 / 10

✓ - **0 pts** Correct

- **2 pts** Some minor arithmetic or other computational errors
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$$\bullet \begin{cases} h(B) = 1 \\ h(x) = 0, \forall x \in \mathcal{C} \setminus \{B\} \\ h(x) = \sum_{y \in \mathcal{S}} p(x, y)h(y), \forall x \in \mathcal{S} \setminus \mathcal{C} \end{cases} \Rightarrow h((i, j)) = \begin{bmatrix} \frac{1}{6} & \frac{4}{15} & \frac{19}{30} & \frac{19}{30} \\ \frac{1}{15} & 0 & 1 & \frac{19}{30} \\ \frac{1}{30} & 0 & 0 & \frac{4}{15} \\ \frac{1}{30} & \frac{1}{30} & \frac{1}{15} & \frac{1}{6} \end{bmatrix}_{i,j}$$

- Therefore,  $\mathbb{P}[X_T = B | X_0 = A] = h(A) = \frac{1}{30}$

### Question 4

**1.56.** A bank classifies loans as paid in full (F), in good standing (G), in arrears (A), or as a bad debt (B). Loans move between the categories according to the following transition probability:

	F	G	A	B
F	1	0	0	0
G	.1	.8	.1	0
A	.1	.4	.4	.1
B	0	0	0	1

What fraction of loans in good standing are eventually paid in full? What is the answer for those in arrears?

- Define  $V_x := \inf\{n \geq 0 | X_n = x\}$ , and  $h(x) := \mathbb{P}_x(V_F < V_G), \forall x \in \mathcal{S}$

$$\bullet \begin{cases} h(G) = 0.1 h(F) + 0.8 h(G) + 0.1 h(A) + 0 h(B) \\ h(A) = 0.1 h(F) + 0.4 h(G) + 0.4 h(A) + 0.1 h(B) \\ h(F) = 1, h(G) = 0 \end{cases} \Rightarrow \begin{cases} h(G) = 0.875 \\ h(A) = 0.75 \end{cases}$$

- So 87.5% of the loans in good standing are eventually paid in full, and 75% for those in arrears

### Question 5

**1.58.** Six children (Dick, Helen, Joni, Mark, Sam, and Tony) play catch. If Dick has the ball he is equally likely to throw it to Helen, Mark, Sam, and Tony. If Helen has the ball she is equally likely to throw it to Dick, Joni, Sam, and Tony. If Sam has the ball he is equally likely to throw it to Dick, Helen, Mark, and Tony. If either Joni or Tony gets the ball, they keep throwing it to each other. If Mark gets the ball he runs away with it. (a) Find the transition probability and classify the states of the chain. (b) Suppose Dick has the ball at the beginning of the game. What is the probability Mark will end up with it?

$$\bullet \mathcal{P} = \begin{matrix} & \begin{matrix} \text{Dick} & \text{Helen} & \text{Joni} & \text{Mark} & \text{Sam} & \text{Tony} \end{matrix} \\ \begin{matrix} \text{Dick} \\ \text{Helen} \\ \text{Joni} \\ \text{Mark} \\ \text{Sam} \\ \text{Tony} \end{matrix} & \begin{bmatrix} 0 & 0.25 & 0 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0 & 0.25 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0.25 & 0.25 & 0 & 0.25 & 0 & 0.25 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

- State  $J, M, T$  are recurrent, since  $\{J, T\}$  and  $\{M\}$  are two closed recurrent sets



3 Problem 3 10 / 10

✓ - 0 pts Correct

- 10 pts No work submitted

$$\bullet \begin{cases} h(B) = 1 \\ h(x) = 0, \forall x \in \mathcal{C} \setminus \{B\} \\ h(x) = \sum_{y \in \mathcal{S}} p(x, y)h(y), \forall x \in \mathcal{S} \setminus \mathcal{C} \end{cases} \Rightarrow h((i, j)) = \begin{bmatrix} \frac{1}{6} & \frac{4}{15} & \frac{19}{30} & \frac{19}{30} \\ \frac{1}{15} & 0 & 1 & \frac{19}{30} \\ \frac{1}{30} & 0 & 0 & \frac{4}{15} \\ \frac{1}{30} & \frac{1}{30} & \frac{1}{15} & \frac{1}{6} \end{bmatrix}_{i,j}$$

- Therefore,  $\mathbb{P}[X_T = B | X_0 = A] = h(A) = \frac{1}{30}$

### Question 4

**1.56.** A bank classifies loans as paid in full (F), in good standing (G), in arrears (A), or as a bad debt (B). Loans move between the categories according to the following transition probability:

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What fraction of loans in good standing are eventually paid in full? What is the answer for those in arrears?

- Define  $V_x := \inf\{n \geq 0 | X_n = x\}$ , and  $h(x) := \mathbb{P}_x(V_F < V_G), \forall x \in \mathcal{S}$

$$\bullet \begin{cases} h(G) = 0.1 h(F) + 0.8 h(G) + 0.1 h(A) + 0 h(B) \\ h(A) = 0.1 h(F) + 0.4 h(G) + 0.4 h(A) + 0.1 h(B) \\ h(F) = 1, h(G) = 0 \end{cases} \Rightarrow \begin{cases} h(G) = 0.875 \\ h(A) = 0.75 \end{cases}$$

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### Question 5

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- State  $J, M, T$  are recurrent, since  $\{J, T\}$  and  $\{M\}$  are two closed recurrent sets

#### 4 Problem 4: Durrett 1.56 10 / 10

✓ - 0 pts Correct

- 2 pts Some minor computational errors

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- So 87.5% of the loans in good standing are eventually paid in full, and 75% for those in arrears

### Question 5

**1.58.** Six children (Dick, Helen, Joni, Mark, Sam, and Tony) play catch. If Dick has the ball he is equally likely to throw it to Helen, Mark, Sam, and Tony. If Helen has the ball she is equally likely to throw it to Dick, Joni, Sam, and Tony. If Sam has the ball he is equally likely to throw it to Dick, Helen, Mark, and Tony. If either Joni or Tony gets the ball, they keep throwing it to each other. If Mark gets the ball he runs away with it. (a) Find the transition probability and classify the states of the chain. (b) Suppose Dick has the ball at the beginning of the game. What is the probability Mark will end up with it?

$$\bullet \mathcal{P} = \begin{matrix} & \begin{matrix} \text{Dick} & \text{Helen} & \text{Joni} & \text{Mark} & \text{Sam} & \text{Tony} \end{matrix} \\ \begin{matrix} \text{Dick} \\ \text{Helen} \\ \text{Joni} \\ \text{Mark} \\ \text{Sam} \\ \text{Tony} \end{matrix} & \begin{bmatrix} 0 & 0.25 & 0 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0 & 0.25 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0.25 & 0.25 & 0 & 0.25 & 0 & 0.25 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

- State  $J, M, T$  are recurrent, since  $\{J, T\}$  and  $\{M\}$  are two closed recurrent sets

- State  $D, H, S$  are transient, since they all communicate with either  $J, M$ , or  $T$
- Define  $V_M := \inf\{n \geq 0 | X_n = M\}$ ,  $V_{J,T} := \inf\{n \geq 0 | X_n \in \{J, T\}\}$ ,  $h(x) := \mathbb{P}_x(V_M < V_{J,T})$
- $$\begin{cases} h(M) = 1, h(J) = 0, h(T) = 0 \\ h(D) = 0.25 h(H) + 0.25 h(M) + 0.25 h(S) + 0.25 h(T) \\ h(H) = 0.25 h(D) + 0.25 h(J) + 0.25 h(S) + 0.25 h(T) \\ h(S) = 0.25 h(D) + 0.25 h(H) + 0.25 h(M) + 0.25 h(T) \end{cases} \Rightarrow \begin{cases} h(D) = 0.4 \\ h(H) = 0.2 \\ h(S) = 0.4 \end{cases}$$
- Therefore the probability Mark will end up with it is 0.4

## Question 6

**1.67.** Roll a fair die repeatedly and let  $Y_1, Y_2, \dots$  be the resulting numbers. Let  $X_n = |\{Y_1, Y_2, \dots, Y_n\}|$  be the number of values we have seen in the first  $n$  rolls for  $n \geq 1$  and set  $X_0 = 0$ .  $X_n$  is a Markov chain. (a) Find its transition probability. (b) Let  $T = \min\{n : X_n = 6\}$  be the number of trials we need to see all 6 numbers at least once. Find  $ET$ .

- $$p(i, j) = \begin{cases} i/6 & j = i \\ 1 - i/6 & j = i + 1 \\ 0 & \text{o.w.} \end{cases}$$
- Define  $g(x) = \mathbb{E}_x[T]$ , then we want to find  $g(0)$
- $$\begin{cases} g(0) = 1 + g(1) \\ g(1) = 1 + \frac{1}{6}g(1) + \frac{5}{6}g(2) \\ g(2) = 1 + \frac{2}{6}g(2) + \frac{4}{6}g(3) \\ g(3) = 1 + \frac{3}{6}g(3) + \frac{3}{6}g(4) \\ g(4) = 1 + \frac{4}{6}g(4) + \frac{2}{6}g(5) \\ g(5) = 1 + \frac{5}{6}g(5) \end{cases} \Rightarrow \begin{cases} g(0) = 14.7 \\ g(1) = 13.7 \\ g(2) = 12.5 \\ g(3) = 11 \\ g(4) = 9 \\ g(5) = 6 \end{cases}$$
- Therefore  $E[T] = 14.7$

## 5 Problem 5: Durrett 1.58 10 / 10

✓ - 0 pts Correct

- 2 pts (a) There are some mistakes in the transition probabilities
- 2 pts (a) The transient or recurrent states are not correctly identified
- 2 pts (b) The linear equations are not set up correctly
- 2 pts (b) The linear equations are not solved correctly
- 10 pts No work submitted.

- State  $D, H, S$  are transient, since they all communicate with either  $J, M$ , or  $T$
- Define  $V_M := \inf\{n \geq 0 | X_n = M\}$ ,  $V_{J,T} := \inf\{n \geq 0 | X_n \in \{J, T\}\}$ ,  $h(x) := \mathbb{P}_x(V_M < V_{J,T})$
- $$\begin{cases} h(M) = 1, h(J) = 0, h(T) = 0 \\ h(D) = 0.25 h(H) + 0.25 h(M) + 0.25 h(S) + 0.25 h(T) \\ h(H) = 0.25 h(D) + 0.25 h(J) + 0.25 h(S) + 0.25 h(T) \\ h(S) = 0.25 h(D) + 0.25 h(H) + 0.25 h(M) + 0.25 h(T) \end{cases} \Rightarrow \begin{cases} h(D) = 0.4 \\ h(H) = 0.2 \\ h(S) = 0.4 \end{cases}$$
- Therefore the probability Mark will end up with it is 0.4

## Question 6

**1.67.** Roll a fair die repeatedly and let  $Y_1, Y_2, \dots$  be the resulting numbers. Let  $X_n = |\{Y_1, Y_2, \dots, Y_n\}|$  be the number of values we have seen in the first  $n$  rolls for  $n \geq 1$  and set  $X_0 = 0$ .  $X_n$  is a Markov chain. (a) Find its transition probability. (b) Let  $T = \min\{n : X_n = 6\}$  be the number of trials we need to see all 6 numbers at least once. Find  $ET$ .

- $$p(i, j) = \begin{cases} i/6 & j = i \\ 1 - i/6 & j = i + 1 \\ 0 & \text{o.w.} \end{cases}$$
- Define  $g(x) = \mathbb{E}_x[T]$ , then we want to find  $g(0)$
- $$\begin{cases} g(0) = 1 + g(1) \\ g(1) = 1 + \frac{1}{6}g(1) + \frac{5}{6}g(2) \\ g(2) = 1 + \frac{2}{6}g(2) + \frac{4}{6}g(3) \\ g(3) = 1 + \frac{3}{6}g(3) + \frac{3}{6}g(4) \\ g(4) = 1 + \frac{4}{6}g(4) + \frac{2}{6}g(5) \\ g(5) = 1 + \frac{5}{6}g(5) \end{cases} \Rightarrow \begin{cases} g(0) = 14.7 \\ g(1) = 13.7 \\ g(2) = 12.5 \\ g(3) = 11 \\ g(4) = 9 \\ g(5) = 6 \end{cases}$$
- Therefore  $E[T] = 14.7$

## 6 Problem 6: Durrett 1.67 10 / 10

✓ - 0 pts Correct

- 2 pts (a) There are some mistakes in the transition probabilities
- 2 pts (a) The transient or recurrent states are not correctly identified
- 2 pts (b) The linear equations are not set up correctly
- 2 pts (b) The linear equations are not solved correctly
- 10 pts No work submitted