9/7

Thursday, September 7, 2017

Linear Space / Vector Space

- A set of vectors
- A set of numbers
- Addition of vectors
- Multiply vectors with numbers

Zero Vector

- There is a vector $\mathcal{O}$ such that for all vector x
  - $x + \mathcal{O} = x$
- Theorem
  - If $\mathcal{O}_1$ and $\mathcal{O}_2$ are both zero vectors, then $\mathcal{O}_1 = \mathcal{O}_2$
- Proof
  - $\begin{cases} \mathcal{O}_1 + \mathcal{O}_2 = \mathcal{O}_1 \\ \mathcal{O}_2 + \mathcal{O}_1 = \mathcal{O}_2 \end{cases} \Rightarrow \mathcal{O}_1 = \mathcal{O}_2$

Existence of Negative Vector

- For every vector x, there is a vector y such that
- $x + y = 0$
- denoted as $-x$

Multiplication with Numbers (Scalers)

- x, y: vectors, $\quad s, t$: numbers (Number field: $\mathbb{Q}, \mathbb{R}, \mathbb{C}$)
- $s(x + y) = sx + sy$
- $(s + t)x = sx + tx$
- $s(tx) = (st)x$
- $0 \cdot x = 0$
- $1 \cdot x = x$

Example of a Common Vector Spaces

- $\mathbb{R}^3 = \{(x_1, x_2, x_3) | x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, x_3 \in \mathbb{R}\}$ is a vector space
- Addition and multiplication defined as
  - $(x_1, x_2, x_3) + (y_1, y_2, y_3) \stackrel{\text{def}}{=} (x_1 + y_1, x_2 + y_2, x_3 + y_3)$
  - $t(x_1, x_2, x_3) \stackrel{\text{def}}{=} (tx_1, tx_2, tx_3)$

Example of a Strange Vector Spaces

- Number: $\mathbb{R}$

- Vector: $\mathbb{R}_+ = (0, \infty)$
- Addition
  - $x \oplus y = x \times y$
  - e.g. $\sqrt{2} \oplus \sqrt{2} = \sqrt{2} \times \sqrt{2} = 2$
  - Zero vector: 1
- Inverse of Addition
  - Given x, find y
  - $x \oplus y = 1$
  - $\Rightarrow y = \frac{1}{x}$
- Multiplication with numbers
  - $t \in R,\ x \in R_+$
  - $t \odot x \stackrel{\text{def}}{=} x^t$
- Proof: Distributive law
  - $t \odot (s \odot x) = (x^s)^t = x^{st} = (ts) \odot x$

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Monday, September 11, 2017

Field

- A field $\mathbb{F}$ is a set together with 2 binary operations
- $+, \times$ ($-$ optional) that satisfies the following:
  - $a + b = b + a$
  - $(a + b) + c = a + (b + c)$
  - $a \times b = b \times a$
  - $(a \times b) \times c = a \times (b \times c)$
  - $a \times (b + c) = a \times b + a \times c$
  - There is a special element $\mathcal{O}$, such that $a + \mathcal{O} = a$
  - There is a special element 1, such that $1 \times a = a$
  - For all a, there is a b, such that $a + b = 0$
  - For any a $\neq \mathcal{O}$, there is a b, such that $a \times b = 1$
  - Optional: $1 \neq \mathcal{O}, \qquad \mathcal{O} \neq 1$
- Example
  - $\mathbb{F} = \{0,1\}$
  - $+ := \begin{cases} 0 + 0 = 0 \\ 0 + 1 = 1 \\ 1 + 1 = 0 \end{cases}$
  - $\times := \begin{cases} 0 \times 0 = 0 \\ 0 \times 1 = 0 \\ 1 \times 1 = 1 \end{cases}$
- Example
  - $\mathbb{F} = \{0,1,2\}$
  - $+ := \begin{cases} 0 + 0 = 0 \\ 0 + 1 = 1 \\ 0 + 2 = 2 \\ 1 + 1 = 2 \\ 1 + 2 = 0 \\ 2 + 2 = 1 \end{cases}$
  - $\times := \begin{cases} 0 \times 0 = 0 \\ 0 \times 1 = 0 \\ 0 \times 2 = 0 \\ 1 \times 1 = 1 \\ 1 \times 2 = 2 \\ 2 \times 2 = 1 \end{cases}$

Vector Space

- A vector space V (over $\mathbb{F}$) is a set together with binary operations

- $\begin{cases} +: V + V \to V \\ \times: F \times V \to V \end{cases}$, such that
  - $\mathbb{F}$ is a field
  - $u + v = v + u, \qquad \forall u, v \in V$
  - $(u + v) + w = v + (u + w), \qquad \forall u, v, w \in V$
  - There is a 0 and vector $\vec{0}$, such that
    - $\forall u, v \in V, \qquad \forall a, b \in \mathbb{F}$
    - $u + \vec{0} = u$
    - $0 \times u = \vec{0}$
    - $a \times \vec{0} = \vec{0}$
    - $(a \times b) \times u = a \times (b \times u)$
    - $(a + b) \times u = a \times u + b \times u$
    - $a(u + v) = a \times u + a \times v$
    - $u + (-1)u = \big(1 + (-1)\big) \times u = 0 \times u = \vec{0}$

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Tuesday, September 12, 2017

What does a proof look like?

- Assumptions
- Conclusion
- Proof

Example 1

- Assumption
  - $V = \{(x_1, x_2, x_3) | x_1, x_2, x_3 \in \mathbb{R} \text{ and } x_1 + x_3 = 0\}$
  - $\forall\, x, y \in V, x + y$ is defined by
  - $z = x + y$ if $z = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$
  - tx is defined by $tx = (tx_1, tx_2, tx_3)$ for every $x \in V, t \in \mathbb{R}$
- Conclusion
  - V is a vector space
- Proof: Axiom 1 ($\forall x, y \in V: x + y \in V$)
  - $let\ z = (z_1, z_2, z_3) = x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$
  - $z_1 + z_3 = x_1 + y_1 + x_3 + y_3 = (x_1 + x_3) + (z_1 + z_3) = 0$
  - $\Rightarrow z \in V$

Example 2

- Assumption
  - $V = \{(x_1, x_2, x_3) | x_1, x_2, x_3 \in \mathbb{R} \text{ and } x_1 + x_3 = 1\}$
  - $\forall\, x, y \in V, x + y$ is defined by
  - $z = x + y$ if $z = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$
  - tx is defined by $tx = (tx_1, tx_2, tx_3)$ for every $x \in V, t \in \mathbb{R}$
- Conclusion
  - V is not a vector Space
- Proof: $\exists x, y \in V: x + y \notin V$

Axiom 5

- To show Axiom 5 does not hold,
- we have to prove for every $\mathcal{O} \in V$,
- there is an $x \in V$ with $\mathcal{O} + x \neq x$

Example 3

- Assumption

- $V = \{\text{all functions } f: [0,1] \to R\}$
- Conclusion
  - V is a vector space
- Proof: Axiom 3 $(\forall f, g \in V: f + g = g + f)$
  - Let $h = f + g$ and $k = g + f$
  - Both h and g has a domain of $[0,1]$
  - $h(x) = f(x) + g(x) = g(x) + f(x) = k(x)$

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Wednesday, September 13, 2017

How to Check Vector Space

- Check 10 axioms
- Check that it's a nonempty subset of a vector space and closed under addition and scalar multiplication
- (By Theorem 1.4, this is enough)

1.6 Subspaces of a linear space

Given a linear space V, let S be a nonempty subset of V. If S is also a linear space, with the same operations of addition and multiplication by scalars, then S is called a *subspace* of V. The next theorem gives a simple criterion for determining whether or not a subset of a linear space is a subspace.

THEOREM 1.4. *Let S be a nonempty subset of a linear space V. Then S is a subspace if and only if S satisfies the closure axioms.*

Proof. If S is a subspace, it satisfies all the axioms for a linear space, and hence, in particular, it satisfies the closure axioms.

Now we show that if S satisfies the closure axioms it satisfies the others as well. The commutative and associative laws for addition (Axioms 3 and 4) and the axioms for multiplication by scalars (Axioms 7 through 10) are automatically satisfied in S because they hold for all elements of V. It remains to verify Axioms 5 and 6, the existence of a zero element in S, and the existence of a negative for each element in S.

Let x be any element of S. (S has at least one element since S is not empty.) By Axiom 2, ax is in S for every scalar a. Taking $a = 0$, it follows that $0x$ is in S. But $0x = O$, by Theorem 1.3(a), so $O \in S$, and Axiom 5 is satisfied. Taking $a = -1$, we see that $(-1)x$ is in S. But $x + (-1)x = O$ since both x and $(-1)x$ are in V, so Axiom 6 is satisfied in S. Therefore S is a subspace of V.

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Thursday, September 14, 2017

Subspace

- Theorem
  - V: vector space
  - S: a subset of V ($S \subseteq V$)
  - If for every $x, y \in S$, we have $x + y \in S$
  - And if for every $x \in S, t \in \mathbb{R}$, we have $tx \in S$
  - Then S is also a vector space
- Given
  - It has been shown that
  - $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) | x_1, x_2, \ldots, x_n \in \mathbb{R}\}$
  - is a vector space
- Example
  - Is $S = \{(x_1, x_2, x_3) | 2x_2 + x_2 = 0\}$ a vector space?
  - $S \in \mathbb{R}^3$, so we only need to verify the closure axioms
    - $x, y \in S \Rightarrow x + y \in S$
    - $x \in S, t \in \mathbb{R} \Rightarrow tx \in S$
- Linear subspace
  - If V is a vector space and $S \subseteq V$ is also a vector space,
  - then S is called a linear subspace of V
- Function space example 1
  - $V = \{\text{all real} - \text{valued functions with domain } [0,1]\}$
  - $= \{f | f: [0,1] \to R\}$ is a vector space
- Function space example 2
  - $(x_1, x_2, \ldots, x_n)$ could be viewed as a function
  - from the set $\{1,2,3, \ldots, n\}$ to $\mathbb{R}$

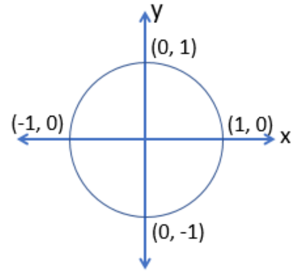
Span of Vector Spaces

- Linear Combination
  - Given
    - V is a vector space
    - $v_1, v_2, \ldots, v_n \in V$
    - $c_1, c_2, \ldots, c_n \in \mathbb{R}$

- then $c_1v_1 + c_2v_2 + \cdots + c_nv_n$ is called
- a linear combination of $v_1, v_2, \dots, v_n$

- Span
  - If V is a vector space and $A \subseteq V$ is a subspace of V
  - then the span of A is the set of all linear combinition of vectors in A
  - $\text{span}(A) = \left\{ c_1v_1 + c_2v_2 + \cdots + c_nv_n \middle| \begin{matrix} v_1, v_2, \dots, v_n \in S \\ c_1, c_2, \dots, c_n \in \mathbb{R} \end{matrix}, n \geq 1 \right\}$

- Example
  - $V = \mathbb{R}^2$
  - $A = \{(x_1, x_2) | x_1^2 + x_2^2 \leq 1\}$
  - $\text{span(A)} = \mathbb{R}^2$



Span of Function spaces

- Example
  - V ={all real-valued functions with domain $[-\pi, +\pi]$}
  - $A = \{1, x, x^2, x^3, x^4\}$
  - Span of A contains function of the form
    - $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$
    - where $a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}$
  - $\Rightarrow \text{span}(A) = \{\text{all polynomials of degree} \leq 4 \text{ with domain } [-\pi, +\pi]\}$

- Change of Domain
  - V ={all real-valued functions with domain $\{0,1\}$}
  - $A = \{1, x, x^2, x^3, x^4\}$
  - $\text{span}(A) = \{x\}$

- Question
  - Does $x^5 \in \text{span}\{1, x, x^2, x^3, x^4\}$ with domain $[-\pi, +\pi]$
  - No, suppose $x^5 \in \text{span}\{1, x, x^2, x^3, x^4\}$, then
    - $x^5 = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4, \quad (\forall x \in [-\pi, +\pi])$
    - Let $x = 0 \Rightarrow a_0 = 0$
  - Differentiate both side, we get
    - $5x^4 = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3$
    - Let $x = 0 \Rightarrow a_1 = 0$
  - Differentiate both side, we get
    - $4 \cdot 5x^3 = 2a_2 + 6a_3x + 12a_4x^2$
    - Let $x = 0 \Rightarrow a_2 = 0$
  - Similarly

        - $a_0 = a_1 = a_2 = a_3 = a_4$
        - $\Rightarrow x^5 = 0$, $(\forall x \in [-\pi, +\pi])$
    - Let $x = 1$, we get $1^5 = 1 = 0$
    - Therefore x^5 is not in $\text{span}\{1, x, x^2, x^3, x^4\}$

Linear Dependence

- ### Definition
    - If V is a vector space, $v_1, \dots, v_n \in V$
    - $\{v_1, \dots, v_n\}$ are linearly independent if for every $c_1, \dots, c_n \in \mathbb{R}$
        - $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$
    - We have
        - $c_1 = c_2 = \cdots = c_n = 0$
    - i.e. The only linear combination of $\{v_1, \dots, v_n\}$ that adds up to 0 is
        - $0v_1 + 0v_2 + \cdots + 0v_n = 0$
- ### Example 1
    - $v_1 = (1,0), \quad v_2 = (0,1), \quad v_3 = (2,2)$
    - $\{v_1, v_2, v_3\}$ is linear dependent ,because $2v_1 + 2v_2 - v_3 = 0$
- ### Example 2
    - $v = \{0\}$ is linear dependent, because $2 \times 0 = 0$

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Monday, September 18, 2017

Question 1

- Let V be a vector space, $S \subseteq T \subseteq V$ be subsets
- Prove or disprove:
- S independence $\Rightarrow$ T independence
  - False
  - Counterexample 1
    - $V = \{0\}$
    - $T = \{0\}$
    - $S = \emptyset$
  - Counterexample 2
    - $V = \mathbb{R}^2$
    - $T = \{(0,1), (1,0), (1,1)\}$
    - $S = \{(0,1), (1,0)\}$
- T independence $\Rightarrow$ S independence
  - True
- $\text{span}(S) = V \Rightarrow \text{span}(T) = V$
  - True
- $\text{span}(T) = V \Rightarrow \text{span}(S) = V$
  - False
  - Counterexample
    - $\mathrm{V} = \mathbb{R}^3$
    - $\mathrm{T} = \{(1,0,0), (0,1,0), (0,0,1)\}$
    - $S = \{(1,0,0)\}$

Question 2

- For which functions f: $\mathbb{R} \to \mathbb{R}$ is $\{f, f'\}$ linear dependent
- $f(x) = Ae^{tx}$ where $k \neq 1$ and $A \neq 0$

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Tuesday, September 19, 2017

Linear Independence

- If $v_1, \dots, v_n \in V$ then $\{v_1, \dots, v_n\}$ is linearly independent
- if for every $c_1, \dots, c_n \in \mathbb{R}$ it follows from
- $c_1 v_1 + \cdots + c_n v_n = 0$ that $c_1 = \cdots = c_n = 0$

Basis

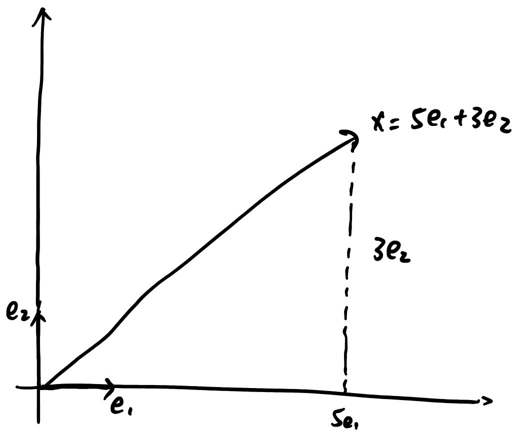
- Definition
  - $\{v_1, \dots, v_n\}$ is a basis for V if
    - $\{v_1, \dots, v_n\}$ are linearly independet
    - Every $x \in V$ is a linear combination of $\{v_1, \dots, v_n\}$
  - i.e. $x = c_1 v_1 + \cdots + c_n v_n \in V$ for certain $c_1, \dots, c_n$
- Theorem
  - If $\{v_1, \dots, v_n\}$ is a basis for V
  - and $\{w_1, \dots, w_m\}$ is also a basis for V
  - Then $n = m$
- Example of no basis
  - $V = \{\text{all function } f: [0,1] \to \mathbb{R}\}$ have no basis
  - If V have no basis then V is called infinite dimensional
- Example of basis $\emptyset$
  - $\emptyset$ is a basis for $\mathrm{V} = \{0\}$
- Dimension
  - If $\{v_1, \dots, v_n\}$ is a basis for V
  - Then n is the dimension of V
- Example 1
  - $V = \mathbb{R}^2, \qquad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
  - Conclusion
    - $\{e_1, e_2\}$ is a basis for $\mathbb{R}^2$
  - Proof: $\{e_1, e_2\}$ is independent
    - Suppose $c_1, c_2 \in \mathbb{R}$ with $c_1 e_1 + c_2 e_2 = 0$
    - Then $c_1 e_1 + c_2 e_2 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
    - Hence $c_1 = c_2 = 0$
  - Proof: $\{e_1, e_2\}$ spans $\{\mathbb{R}^2\}$

        - Given $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$
        - We can find c_1, c_2 such that $x = c_1 e_1 + c_2 e_2$
        - $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$
        - Choose $\begin{cases} c_1 = x_1 \\ c_2 = x_2 \end{cases}$,
        - Therefore the basis spans $\mathbb{R}^2$
- Example 2
    - $V = \mathbb{R}^2, \qquad e_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
    - Conclusion
        - $\{e_1, e_2\}$ is a basis for $\mathbb{R}^2$
    - Proof: $\{e_1, e_2\}$ is independent
        - Trivial
    - Proof: $\{e_1, e_2\}$ spans $\{\mathbb{R}^2\}$
        - Given $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$
        - $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2c_1 + c_2 \\ c_1 + c_2 \end{pmatrix}$
        - Choose $\begin{cases} c_1 = x_1 + x_2 \\ c_2 = -x_1 + 2x_2 \end{cases}$
        - Therefore the basis spans $\mathbb{R}^2$
- Theorem
    - Statement
        - If $\{e_1, .., e_n\}$ are linearly independent and if
        - $c_1 e_1 + \cdots + c_n e_n = b_1 e_1 + \cdots + b_n e_n$
        - for certain $c_1, \ldots, c_n, b_1, \ldots, b_n$
        - then $c_1 = b_1, c_2 = b_2, \ldots, c_n = b_n$
    - Proof
        - $c_1 e_1 + \cdots + c_n e_n = b_1 e_1 + \cdots + b_n e_n$
        - $(c_1 - b_1) e_1 + \cdots + (c_n - b_n) e_n = 0$
        - $\{e_1, .., e_n\}$ are linearly independent
        - $\Rightarrow c_1 - b_1 = 0, \ldots, c_n - b_n = 0$
        - $\Rightarrow c_1 = b_1, \ldots, c_n = b_n$

Coordinates / Components

- Theorem
    - If $\{e_1, .., e_n\}$ is a basis for vector space V
    - Then for every $x \in V$, there is a unique choice of

- $c_1, c_2, \dots, c_n \in R$ with $x = c_1 e_1 + \cdots + c_n e_n$
- $c_1, \dots, c_n$ are called the coordinates or components of x



- Example
  - $V = \{\text{all function } f: \mathbb{R} \to \mathbb{R}\}$
  - $W = \{\text{all } f \in V \text{ that satisfy } f'' = f\}$
  - Given
    - $f(x) = e^x \in W$
    - $g(x) = e^{-x} \in W$
    - $h(x) = sinh\, x \in W$
    - $f(x) = cosh\, x \in W$
  - Are f, g, h, k linear independent?
    - No, because
    - $\frac{1}{2}e^x - \frac{1}{2}e^{-1} - sinh\, x = 0$

9/20

Wednesday, September 20, 2017

Question 1

- Why is span(∅) = {0}?
- 0 is the additive identity

Question 2

- The basis of $V = \left\{f \in P_n \middle| f(0) + f'(0) = 0\right\}$?
- $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$
- $f(0) = a_0, \qquad f'(0) = a_1$
- $f(0) + f'(0) = 0$
- $\Rightarrow a_0 = -a_1$
- $f(x) = a_1(x - 1) + a_2 x^2 + \cdots + a_n x^n$
- Therefore the basis of V is $\{x - 1, x^2, \dots, x^n\}$

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Thursday, September 21, 2017

Theorem 1.5

THEOREM 1.5. *Let $S = \{x_1, \ldots, x_k\}$ be an independent set consisting of k elements in a linear space V and let $L(S)$ be the subspace spanned by S. Then every set of $k + 1$ elements in $L(S)$ is dependent.*

Proof. The proof is by induction on k, the number of elements in S. First suppose $k = 1$. Then, by hypothesis, S consists of one element x_1, where $x_1 \neq O$ since S is independent. Now take any two distinct elements y_1 and y_2 in $L(S)$. Then each is a scalar multiple of x_1, say $y_1 = c_1x_1$ and $y_2 = c_2x_1$, where c_1 and c_2 are not both 0. Multiplying y_1 by c_2 and y_2 by c_1 and subtracting, we find that

$$c_2y_1 - c_1y_2 = O.$$

This is a nontrivial representation of O so y_1 and y_2 are dependent. This proves the theorem when $k = 1$.

Now we assume the theorem is true for $k - 1$ and prove that it is also true for k. Take any set of $k + 1$ elements in $L(S)$, say $T = \{y_1, y_2, \ldots, y_{k+1}\}$. We wish to prove that T is dependent. Since each y_i is in $L(S)$ we may write

$$y_i = \sum_{j=1}^{k} a_{ij}x_j \tag{1.4}$$

for each $i = 1, 2, \ldots, k + 1$. We examine all the scalars a_{i1} that multiply x_1 and split the proof into two cases according to whether all these scalars are 0 or not.

CASE 1. $a_{i1} = 0$ for every $i = 1, 2, \ldots, k + 1$. In this case the sum in (1.4) does not involve x_1, so each y_i in T is in the linear span of the set $S' = \{x_2, \ldots, x_k\}$. But S' is independent and consists of $k - 1$ elements. By the induction hypothesis, the theorem is true for $k - 1$ so the set T is dependent. This proves the theorem in Case 1.

CASE 2. Not all the scalars a_{i1} are zero. Let us assume that $a_{11} \neq 0$. (If necessary, we can renumber the y's to achieve this.) Taking $i = 1$ in Equation (1.4) and multiplying both members by c_i, where $c_i = a_{i1}/a_{11}$, we get

$$c_iy_1 = a_{i1}x_1 + \sum_{j=2}^{k} c_ia_{1j}x_j.$$

From this we subtract Equation (1.4) to get

$$c_iy_1 - y_i = \sum_{j=2}^{k}(c_ia_{1j} - a_{ij})x_j,$$

for $i = 2, \ldots, k + 1$. This equation expresses each of the k elements $c_iy_1 - y_i$ as a linear combination of the $k - 1$ independent elements $x_2, \ldots, x_k$. By the induction hypothesis, the k elements $c_iy_1 - y_i$ must be dependent. Hence, for some choice of scalars $t_2, \ldots, t_{k+1}$, not all zero, we have

$$\sum_{i=2}^{k+1} t_i(c_iy_1 - y_i) = O,$$

from which we find

$$\left(\sum_{i=2}^{k+1} t_ic_i\right) y_1 - \sum_{i=2}^{k+1} t_iy_i = O.$$

But this is a nontrivial linear combination of $y_1, \ldots, y_{k+1}$ which represents the zero element, so the elements $y_1, \ldots, y_{k+1}$ must be dependent. This completes the proof.

Theorem 1.6

- Statement
  - If $\{v_1 \dots v_n\}$ and $\{w_1, \dots, w_m\}$ are bases for V, then $n = m$
- Proof
  - Suppose $n < m$
  - $w_1, \dots, w_m, w_{m+1} \in span\{v_1 \dots v_n\}$
  - $\Rightarrow \{w_1, \dots, w_n, w_{n+1}\}$ are linearly dependent by previous therom
  - $\Rightarrow \{w_1, \dots, w_n, w_{n+1}, \dots, w_m\}$ are also linearly dependent
  - But $\{w_1, \dots, w_m\}$ is linearly independet, because it a basis for V
  - So $n < m$ is not true
  - Similarly the assumption $n > m$ also leads to contradiction
  - Therefore $n = m$
- Example
  - Given
    - $f(x) = 1 + 2x + x^2$
    - $g(x) = x^2 - 4$
    - $h(x) = 2x - x^2$
    - $k(x) = x - 3$
  - Claim
    - There exist $c_1, c_2, c_3, c_4 \in \mathbb{R}$
    - such that $c_1 f(x) + c_2 g(x) + c_3 h(x) + c_4 k(x) = 0$
    - And at least one of c_1, c_2, c_3, c_4 is not 0
    - $V = \{\text{all polynomials of degree} \leq 2\}$ has basis $\{1, x, x^2\}$
    - $f, g, h, k \in \text{span}\{1, x, x^2\}$
    - $\Rightarrow$ f, g, h, k are linearly dependent

Theorem

- Statement
  - If V is a n-dimensional vector space
  - And $v_1, \dots, v_m \in V$ are linearly independence with $m < n$
  - Then there exist $v_{m+1}, \dots, v_n \in V$
  - Such that $\{v_1, \dots, v_n\}$ is a basis for V
- Outline of proof
  - $\text{span}\{v_1, \dots, v_m\} \neq V$ by the previous theorem
  - Choose $v_{m+1} \in V$ such that $v_m \notin \text{span}\{v_1, \dots, v_m\}$
  - Then $\{v_1, \dots, v_m, v_{m+1}\}$ is also linearly independent
  - If $m + 1 = n$, then $\{v_1, \dots, v_m, v_{m+1}\}$ is a basis for V

- Or $m + 1 < n$, then repeat the previous steps

9/25

Monday, September 25, 2017

Span

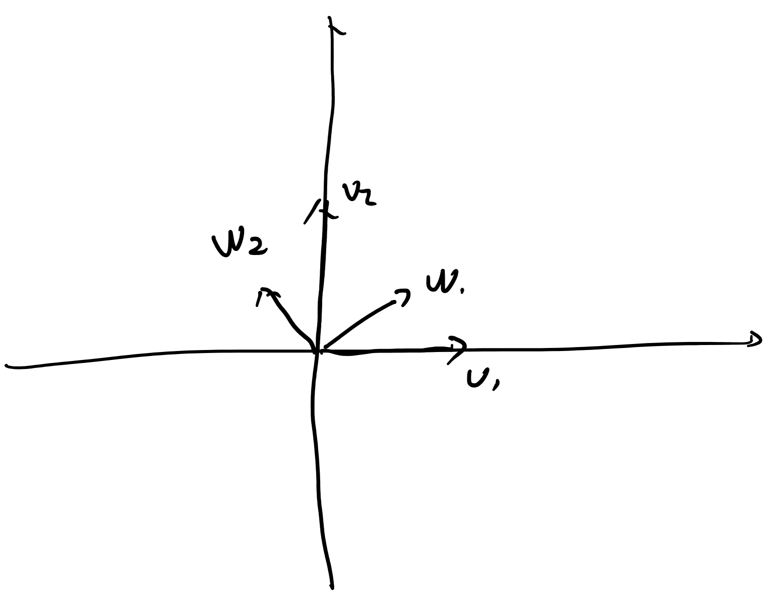
- $L(S) = \left\{ x \in V \;\middle|\; \begin{matrix} \exists n \in \mathbb{N} \\ \exists c_1, \dots, c_n \in \mathbb{R} \\ \exists x_1, \dots, x_n \in S \\ x = c_1 x_1 + \cdots + c_n x_n \end{matrix} \right\}$

Theorem

- Statement
  - $S \subseteq V$ is a subspace $\Leftrightarrow S = L(S)$
- Proof: $S = L(S) \Rightarrow S \subseteq V$ is a subspace
  - Let $s, t \in S,\ k \in \mathbb{R}$
  - Then $s + k \cdot t \in L(S)$
  - $L(S) = S \Rightarrow s + k \cdot t \in S$
  - $\Rightarrow S$ is closed under addition and scalar multiplication
  - Therefore S is a subspace of V
- Proof: $S \subseteq V$ is a subspace $\Rightarrow S = L(S)$
  - If $T \subseteq V$ and T is a subspace, then $L(S) \subseteq T$
  - Setting $T = S$, we have $L(S) \subseteq S$
  - We also know that $S \subseteq L(S)$
  - So $S = L(S)$ by definition of set equality

Question 1

- Example of $L(S \cap T) \neq L(S) \cap L(T)$, where $S, T \subseteq V$



  - $V = \mathbb{R}^2$

- $\mathrm{S} = \{v_1, v_2\}, T = \{w_1, w_2\}$
- $\mathrm{L}(\mathrm{S} \cap \mathrm{T}) = \mathrm{L}(\emptyset) = \{0\}$
- $\mathrm{L}(\mathrm{S}) = \mathrm{L}(\mathrm{R}) = \mathbb{R}^2$

Question 2

- Let $S_1, \dots, S_n$ be subsets of V
- When is $L(S_1) \cup \cdots \cup L(S_n)$ a subspace?
- $L(S_1) \cup L(S_2)$ is a subspace $\Leftrightarrow L(S_1) \subseteq L(S_2)$ or $L(S_2) \subseteq L(S_1)$

9/26

Tuesday, September 26, 2017

Inner Product

- Definition (on real vector space)
  - An inner product on a real vector space V
  - is a real-valued function (x, y) with $x, y \in V$
  - for which:
    - $(x + y, z) = (x, z) + (y, z), \qquad \forall x, y, z \in V$
    - $(tx, y) = t(x, y), \qquad \forall x, y \in V$, and $t \in \mathbb{R}$
    - $(x, y) = (y, x), \qquad \forall x, y \in V$
    - $(x, x) \geq 0, \qquad \forall x \in V$
    - $(x, x) = 0 \Rightarrow x = 0$
- Definition (on complex vector space)
  - An inner product on a real vector space V
  - is a real-valued function (x, y) with $x, y \in V$
  - for which:
    - $(x + y, z) = (x, z) + (y, z), \qquad \forall x, y, z \in V$
    - $(tx, y) = t(x, y), \qquad \forall x, y \in V$, and $t \in \mathbb{R}$
    - $(x, y) = \overline{(y, x)}, \qquad \forall x, y \in V$
    - $(x, x) \geq 0, \qquad \forall x \in V$
    - $(x, x) = 0 \Rightarrow x = 0$
  - Note: $(x, ty) = \overline{(ty, x)} = \bar{t}(x, y)$
- Example in $\mathbb{R}^2$
  - Let $V = \mathbb{R}^2$
  - The following is an inner product for V
    - $(x, y) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$
  - Proof: $(tx, y) = t(x, y)$
    - (tx, y)
    - $= (tx_1) y_1 + (tx_2) y_2 + \cdots + (tx_n) y_n$
    - $= t(x_1 y_1) + t(x_2 y_2) + \cdots + t(x_n y_n)$
    - $= t(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)$
    - $= t(x, y)$
- Example in $\mathbb{C}^n$
  - Let $V = \mathbb{C}^n$

- The following is an inner product for V
  - $(x,y) = x_1\overline{y_1} + x_2\overline{y_2} + \cdots + x_n\overline{y_n}$
- Proof
  - ☑ $(x+y,z) = (x,z) + (y,z)$
  - ☑ $(tx,y) = t(x,y)$
  - ☑ $(x,y) = \overline{(y,x)}$
  - ☑ $(x,x) \geq 0$
  - ☑ $(x,x) = 0 \Rightarrow x = 0$

- Counterexample in $\mathbb{R}^n$
  - Let $V = \mathbb{R}^n$
  - Whether the following is an inner product for V
    - $(x,y) = x_1y_1 - x_2y_2$
  - We need to check
    - ☑ $(x+y,z) = (x,z) + (y,z)$
    - ☑ $(tx,y) = t(x,y)$
    - ☑ $(x,y) = \overline{(y,x)}$
    - ☐ $(x,x) \geq 0$
    - ☐ $(x,x) = 0 \Rightarrow x = 0$
- Counterexample in $\mathbb{R}^n$
  - Let $V = \mathbb{R}^n$
  - Whether the following is an inner product for V
    - $(x,y) = x_1y_1$
  - We need to check
    - ☑ $(x+y,z) = (x,z) + (y,z)$
    - ☑ $(tx,y) = t(x,y)$
    - ☑ $(x,y) = \overline{(y,x)}$
    - ☑ $(x,x) \geq 0$
    - ☐ $(x,x) = 0 \Rightarrow x = 0$
- Example in $\mathbb{R}^{\mathrm{n}}$
  - Let $V = \mathbb{R}^n$
  - The following is an inner product for V
    - $(x,y) = (x_1 + x_2)(y_1 + y_2) + x_2y_2$
- Example in function space
  - $V = C([a,b]) = \{\text{all continuous function on } [a,b]\}$
  - The following is an inner product for V

- $(f,g) = \int_a^b f(x)g(x)dx\,, \qquad \text{where } a < b$

- We need to check
  - ☑ $(f+g,h) = (f,h) + (g,h)$
  - ☑ $(t \cdot f, g) = t(f,g)$
  - ☑ $(f,g) = (g,f)$
  - ☑ $(f,f) \geq 0$
  - ☑ $(f,f) = 0 \Rightarrow f = 0$

Length of Vector

- Definition
  - $\sqrt{(x,x)} = \|x\|$ is called the length of x
  - Note: $(x,x) = \|x\|^2$
- Cauchy Schwarz Inequality
  - $(x,y) \leq |x||y|, \qquad \text{for all } x,y \in V$
  - Proof on page 16

Angle

- Definition
  - If $x,y \in V$ $(x \neq 0, y \neq 0)$
  - Then the angle between x,y is θ where
  - $cos\,\theta = \dfrac{(x,y)}{\|x\| \cdot \|y\|}$
- Note
  - Cauchy Schwarz Inequality implies
  - $-1 \leq \dfrac{(x,y)}{\|x\| \cdot \|y\|} \leq 1$
- Orthogonal
  - Vectors x,y are called orthogonal or perpendicular if
  - $(x,y) = 0$
- Example
  - Given
    - $V = \{\text{all polynomials}\}$
    - $(f,g) = \int_0^1 f(x)g(x)dx$
  - Find the angle θ between $f(x) = 1$ and $g(x) = 1$

- $\|f\| = \sqrt{\int_0^1 f(x)f(x)dx} = \sqrt{\int_0^1 1^2 dx} = 1$
- $\|g\| = \sqrt{\int_0^1 g(x)g(x)dx} = \sqrt{\int_0^1 x^2 dx} = \frac{\sqrt{3}}{3}$
- $(f, g) = \int_0^1 f(x)g(x)dx = \int_0^1 x dx = \frac{1}{2}$
- $cos\,\theta = \frac{(x, y)}{\|x\| \cdot \|y\|} = \frac{\sqrt{3}}{2}$
- $\Rightarrow \theta = \frac{\pi}{6}$

9/27

Wednesday, September 27, 2017

Theorem

- Statement
  - Let $W_1, W_2 \subseteq V$ be subspace
  - $W_1 \cup W_2$ is a subspace $\Leftrightarrow W_1 \subseteq W_2$ or $W_2 \subseteq W_1$
- Proof: $W_1 \subseteq W_2$ or $W_2 \subseteq W_1 \Rightarrow W_1 \cup W_2$ is a subspace
  - Obvious
- Proof: $W_1 \cup W_2$ is a subspace $\Rightarrow W_1 \subseteq W_2$ or $W_2 \subseteq W_1$
  - Suppose
    - $\exists v_1 \in W_1, \quad \text{s.t. } v_1 \notin W_2$
    - $\exists v_2 \in W_2, \quad \text{s.t. } v_2 \notin W_1$
  - Then
    - $v_1 + v_2 \notin W_1$
  - Indeed, if
    - $v_1 + v_2 = w \in W_1$
  - Then
    - $v_2 = w - v_1 \in W_1$
    - Contradiction
  - Likewise
    - $v_1 + v_2 \notin W_2$
  - Therefore
    - $v_1 + v_2 \notin W_1 \cup W_2$

Question 1

- Let V be a vector space, $\langle \cdot, \cdot \rangle$ is an inner product on V
- Prove
  - $\forall\, v, w \in V$
  - $\langle u, v \rangle = 0 \Leftrightarrow \|v + c \cdot w\| \geq \|v\|, \quad \forall c \in R$
- Proof: $\langle u, v \rangle = 0 \Rightarrow \|v + c \cdot w\| \geq \|v\|$
  - $c^2\|w\|^2 \geq 0$
  - $\|v\|^2 + c^2\|w\|^2 \geq \|v\|^2$
  - $\|v\|^2 + 2c\langle u, v \rangle + c^2\|w\|^2 \geq \|v\|^2$
  - $\|v + c \cdot w\|^2 \geq \|v\|^2$
  - $\|v + c \cdot w\| \geq \|v\|$

- Proof: $\|v + c \cdot w\| \geq \|v\| \Rightarrow \langle u, v\rangle = 0$
  - $\|v + c \cdot w\| \geq \|v\|$
  - $\|v\|^2 + 2c\langle u, v\rangle + c^2\|w\|^2 \geq \|v\|^2$
  - In order for the inequality to be true for all c
  - $\langle u, v\rangle = 0$

Question 2

- Let V be a finite-dimensional vector space
- $\langle \cdot, \cdot \rangle$ is an inner product on V
- Let $\mathrm{W} \subseteq \mathrm{V}$ be a subspace
- Define $\mathrm{W}^{\perp} = \{v \in V | \langle v, w\rangle = 0,\ \forall w \in W\}$
- Prove that
  - $\mathrm{W}^{\perp}$ is a subspace
  - $dim\, W + dim\, W^{\perp} = dim\, V$

9/28

Thursday, September 28, 2017

Distance

- Definition
  - Distance between two vectors x, y is defined as
  - $\text{distance}(x, y) = \|x - y\| = \sqrt{(x - y, x - y)}$
- Example 1
  - Given
    - $V = \mathbb{R}^2$
    - $(x, y) = x_1 y_1 + x_2 y_2$
  - Distance between two vectors is
    - $\text{distance}(x, y)$
    - $= \|x - y\|$
    - $= \sqrt{(x - y, x - y)}$
    - $= \sqrt{(x_1 - y_1)^2 - (x_2 - y_2)^2}$
- Example 2
  - Given
    - $V = \{\text{all continuous function } f: [0,1] \to \mathbb{R}\}$
    - $(f, g) = \int_0^1 f(x) g(x) dx$
  - Distance between two functions is
    - $\text{distance}(f, g)$
    - $= \|f - g\|$
    - $= \sqrt{(f - g, f - g)}$
    - $= \int_0^1 (f(x) - g(x))^2 dx$
  - Also known as "root mean square distance"

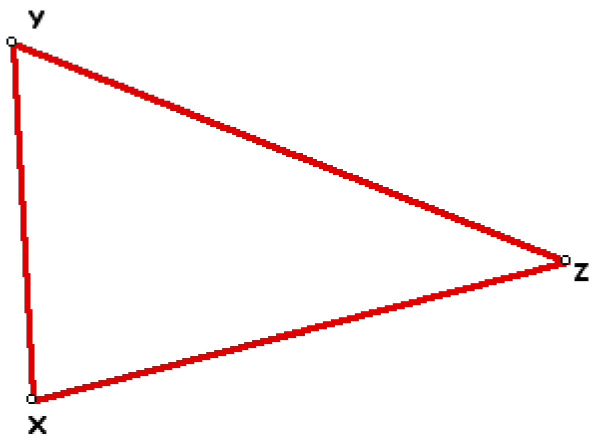
Triangle Inequality (Version 1)

- Statement
  - $\|a + b\| \leq \|a\| + \|b\|$
- Proof
  - $\|a + b\| \stackrel{\text{def}}{=} (a + b, a + b)$

- $= (a, a) + (a, b) + (b, a) + (b, b)$
- $= (a, a) + 2(a, b) + (b, b)$
- $\leq \|a\|^2 + 2\|a\|\|b\| + \|b\|^2$
- $= (\|a\| + \|b\|)^2$
- Therefore $\|a + b\| \leq \|a\| + \|b\|$

Triangle Inequality (Version 2)

- Statement
  - $\text{distance}(x, y) \leq \text{distance}(x, z) + \text{distance}(z, y)$
- Proof
  - Let $a = x - z, \quad b = z - y$
  - then $a + b = x - y$
  - $\|x - y\| \leq \|x - z\| + \|z - y\|$
  - $\text{distance}(x, y) \leq \text{distance}(x, z) + \text{distance}(z, y)$



Orthogonal

- Definition
  - $\{v_1, \dots, v_n\}$ are orthogonal if $(v_k, v_l) = 0, \ \forall k \neq l$
- Theorem
  - If $\{v_1, \dots, v_n\}$ are orthogonal
  - and $v_k \neq 0$ for all $k \in \{1, 2, \dots, n\}$
  - then $\{v_1, \dots, v_n\}$ is linearly independent
- Proof
  - Suppose
    - $c_1 v_1 + \cdots + c_n v_n = 0$
  - Then we have to show
    - $c_1 = c_2 = \cdots = c_n = 0$
  - Let $k \in \{1, 2, \dots, n\}$, then
    - $(c_1 v_1 + \cdots + c_n v_n, v_k) = (0, v_k)$

        - $c_1(v_1, v_k) + \cdots + c_k(v_k, v_k) + \cdots + c_n(v_n, v_k) = 0$
    - Because $(v_k, v_l) = 0,\ \forall k \neq l$, we have
        - $0 + \cdots + 0 + c_k(v_k, v_k) + 0 + \cdots + 0 = 0$
        - $c_k(v_k, v_k) = 0$
    - Because $v_k \neq 0$
        - $(v_k, v_k) \neq 0$
        - $c_k = \dfrac{0}{(v_k, v_k)} = 0$
    - Therefore
        - $c_1 = c_2 = \cdots = c_n = 0$
- Theorem
    - If $x = c_1 v_1 + \cdots + c_n v_n$
    - and $\{v_1, \dots, v_n\}$ are non zero and orthogonal
    - then $c_k = \dfrac{(x, v_k)}{(v_k, v_k)}$
- Proof
    - (x, v_k)
    - $= (c_1 v_1 + \cdots + c_n v_n, v_k)$
    - $= c_1(v_1, v_k) + \cdots + c_k(v_k, v_k) + \cdots + c_n(v_n, v_k)$
    - $= 0 + \cdots + 0 + c_k(v_k, v_k) + 0 + \cdots + 0$
    - $= c_k(v_k, v_k)$
    - $\Rightarrow c_k = \dfrac{(x, v_k)}{(v_k, v_k)}$

Gramm-Schmidt Process

- Introduction
    - If V has a basis $\{v_1, \dots, v_n\}$
    - then there is an orthogonal basis $\{w_1, \dots, w_n\}$
    - The process to find the orthogonal basis is called
    - Gramm-Schmidt Process
- Process
    - $w_1 = v_1$
    - $w_2 = v_2 - \dfrac{(v_2, w_1)}{(w_1, w_1)} w_1$
    - $w_3 = v_3 - \dfrac{(v_3, w_1)}{(w_1, w_1)} w_1 - \dfrac{(v_3, w_2)}{(w_2, w_2)} w_2$
    - $\vdots$

- $w_k = v_k - \sum_{i=0}^{k-1} \frac{(w_k, w_i)}{(w_i, w_i)} w_i$
- Proof: $(w_3, w_2) = 0$
  - Assume we've already shown $(w_1, w_2) = (w_1, w_3) = 0$
  - (w_3, w_2)
  - $= (v_3, w_2) - \frac{(v_3, w_1)}{(w_1, w_1)} \cdot (w_1, w_2) - \frac{(v_3, w_1)}{(w_1, w_1)} \cdot (w_1, w_2)$
  - $= (v_3, w_2) - (v_3, w_2)$
  - $= 0$
- Example 1
  - Given
    - $V = \mathbb{R}^2$
    - $(x, y) = x_1 y_1 + x_2 y_2$
  - Find the orthogonal basis for $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
    - $w_1 = v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
    - $w_2 = v_2 - \frac{(v_2, w_1)}{(w_1, w_1)} w_1 = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}$
    - $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} \right\}$
- Example 2
  - Given
    - $V = \{\text{all continous functions } f: [0,1] \to \mathbb{R}\}$
    - $(f, g) = \int_0^1 f(x) g(x) dx$
  - Find the orthogonal basis for $f_1(x) = 1, f_2(x) = x$
    - $g_1(x) = f_1(x) = 1$
    - $g_2(x) = f_2(x) - \frac{(f_2, g_1)}{(g_1, g_1)} g_1(x) = x - \frac{1}{2}$
    - $\left\{ 1, x - \frac{1}{2} \right\}$