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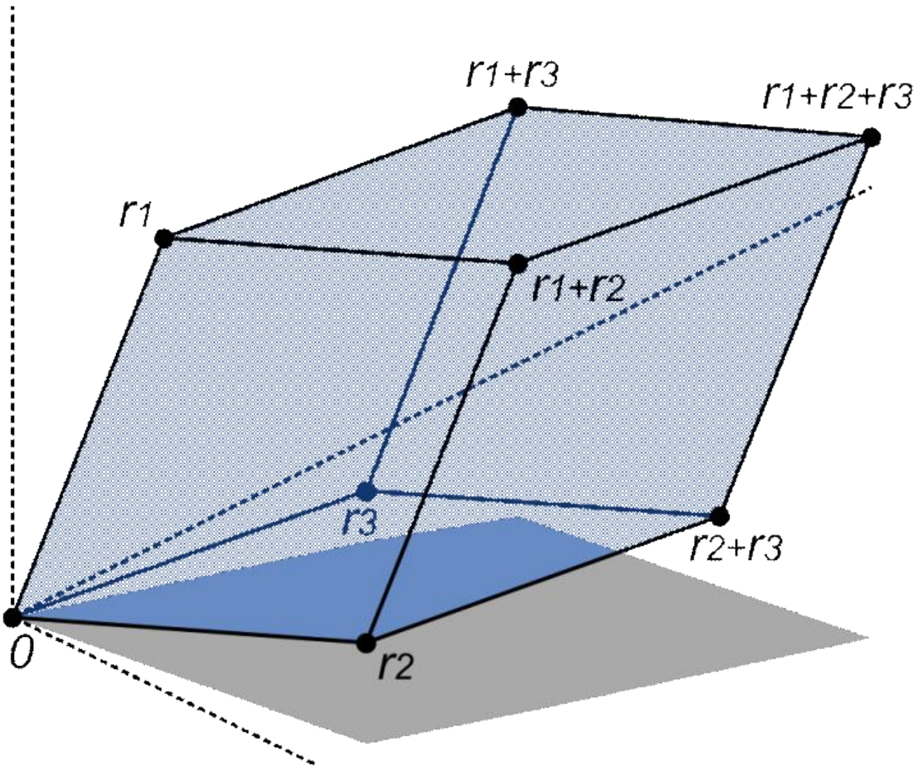
Wednesday, November 1, 2017

Understanding of Determinant in Terms of Volumes



- The volume of this parallelepiped is the absolute value of the determinant of the matrix formed by the rows constructed from the vectors r1, r2, and r3.
- Negative determinant = flip the original image

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Thursday, November 2, 2017

Uniqueness Theorem

- Theorem
  - Suppose $f(A_1, \dots, A_n)$ is a function of $A_1, \dots, A_n \in \mathbb{R}^n$
  - That satisfies Linearity and Alternating
    - $f(B + C, A_2, \dots, A_n) = f(B, A_2, \dots, A_n) + f(C, A_2, \dots, A_n)$
    - $f(t \cdot A_1, A_2, \dots, A_n) = t \cdot f(A_1, A_2, \dots, A_n)$
    - $f\left(A_1, A_2, \dots, A_i, \dots, A_j, \dots A_n\right) = -f\left(A_1, A_2, \dots, A_j, \dots, A_i, \dots A_n\right)$
  - Then $f(A_1, \dots, A_n) = \det(A_1, \dots, A_n) \cdot f(I_1, \dots, I_n)$ where
    - $\mathrm{I}_1 = [1,0,0, \dots, 0]$
    - $\mathrm{I}_2 = [0,1,0, \dots, 0]$
    - $\vdots$
    - $\mathrm{I}_\mathrm{n} = [0,0,0, \dots, 1]$
- Proof
  - $f(A_1, \dots, A_n)$
  - $= f(a_{11}I_1 + a_{12}I_2 + \cdots + a_{1n}I_n, \dots, a_{n1}I_1 + a_{n2}I_2 + \cdots + a_{nn}I_n)$
  - $= \sum_{\substack{1 \le i_1, i_2, \dots, i_n \le n \\ \text{all different}}}^{n} a_{1i_1} a_{2i_2} \dots a_{ni_n} \cdot f\left(I_{i_1}, I_{i_2}, \dots, I_{i_n}\right)$
  - $= \sum_{\substack{1 \le i_1, i_2, \dots, i_n \le n \\ \text{all different}}}^{n} a_{1i_1} a_{2i_2} \dots a_{ni_n} \cdot \mathrm{sign}(i_1, \dots, i_n) \cdot f(I_1, I_2, \dots, I_n)$
  - $= f(I_1, I_2, \dots, I_n) \cdot \sum_{\substack{1 \le i_1, i_2, \dots, i_n \le n \\ \text{all different}}}^{n} a_{1i_1} a_{2i_2} \dots a_{ni_n} \cdot \mathrm{sign}(i_1, \dots, i_n)$
  - $= f(I_1, I_2, \dots, I_n) \cdot \det(A_1, \dots, A_n)$
- Example
  - $\begin{vmatrix} A_{k\times k} & 0 \\ C_{l\times k} & B_{l\times l} \end{vmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{1k} & & & \\ \vdots & \ddots & \vdots & & & \\ a_{k1} & \cdots & a_{kk} & & & \\ c_{11} & \cdots & c_{1k} & b_{11} & \cdots & b_{1l} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{l1} & \cdots & c_{lk} & b_{l1} & \cdots & b_{ll} \end{vmatrix} = \det A \cdot \det B$
  - Consider a function f that satisfies the Uniqueness Theorem
    - $f\left(\overline{A_1} + \overline{\overline{A_1}}, A_2, \dots, A_n\right) = f(\overline{A_1}, A_2, \dots, A_n) + d\left(\overline{\overline{A_1}}, A_2, \dots, A_n\right)$
    - $f(tA_1, A_2, \dots, A_n) = f(A_1, A_2, \dots, A_n)$
    - $f\left(A_1, A_2, \dots, A_i, \dots, A_j, \dots A_n\right) = f\left(A_1, A_2, \dots, A_j, \dots, A_i, \dots A_n\right)$
  - Let $f_{BC}(A_1, \dots, A_k) = \begin{vmatrix} A_{k\times k} & 0 \\ C_{l\times k} & B_{l\times l} \end{vmatrix}$ with B, C fixed, and A as variable

- $f_{BC}(A_1, \dots, A_k)$
- $= \det(A_1, \dots, A_k) f_{BC}(I_1, \dots, I_k)$
- $= \det A \cdot \begin{vmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ c_{11} & \cdots & c_{1k} & b_{11} & \cdots & b_{1l} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{l1} & \cdots & c_{lk} & b_{l1} & \cdots & b_{ll} \end{vmatrix}$
- $= \det A \cdot \begin{vmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & b_{11} & \cdots & b_{1l} \\ & & & \vdots & \ddots & \vdots \\ & & & b_{l1} & \cdots & b_{ll} \end{vmatrix}$
- $= \det A \cdot \begin{vmatrix} I & \\ & B \end{vmatrix}$

  - Let $g(B) = \begin{vmatrix} I & \\ & B \end{vmatrix}$ that satisfies the Uniqueness Theorem
    - $g(B) = \det B \cdot g(I) = \det B \cdot \begin{vmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{vmatrix} = \det B$
  - Therefore $\begin{vmatrix} A_{k\times k} & 0 \\ 0 & B_{l\times l} \end{vmatrix} = \det A \cdot \det B$

Properties of Determinant

- $\det(AB) = \det A \cdot \det B$ (where $A_{n\times n},\ B_{n\times n}$)
  - $\det A \cdot \det B$
  - $= \begin{vmatrix} A & 0 \\ I & B \end{vmatrix}$
  - $= \begin{vmatrix} 0 & -AB \\ I & B \end{vmatrix}$
  - $= (-1)^{n^2} \begin{vmatrix} I & B \\ 0 & -AB \end{vmatrix}$
  - $= (-1)^{n^2} \det I \cdot \det(-AB)$
  - $= (-1)^{n^2} \cdot \det(-AB)$
  - $= (-1)^{n^2} (-1)^n \det(AB)$
  - $= (-1)^{n^2+n} \det(AB)$
  - $= \det(AB)$
- Power of Determinants
  - $\det(A^n) = \det(A \cdot A \dots A) = \det(A) \cdot \det(A) \dots \det(A) = (\det A)^n$
- Determinant of Inverse
  - If A has an inverse(A^{-1}), and $\det A \neq 0$, then
  - $A^{-1}A = I$
  - $\Rightarrow \det A^{-1} \cdot \det A = \det I = 1$
  - $\Rightarrow \det A^{-1} = \dfrac{1}{\det A}$
- Matrix Product and Determinant

- $$\begin{vmatrix} A_{n\times n} & 0 \\ I & B_{n\times n} \end{vmatrix}$$
- $$= \begin{vmatrix} a_{11} & \dots & a_{1n} & & & \\ \vdots & \ddots & \vdots & & & \\ a_{n1} & \dots & a_{nn} & & & \\ 1 & & & b_{11} & \dots & b_{1n} \\ & \ddots & & \vdots & \ddots & \vdots \\ & & 1 & b_{n1} & \dots & b_{nn} \end{vmatrix}$$
- $$= \begin{vmatrix} 0 & \dots & a_{1n} & -a_{11}b_{11} & \dots & -a_{11}b_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & a_{nn} & -a_{n1}b_{11} & \dots & -a_{n1}b_{1n} \\ 1 & \dots & 0 & b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & b_{n1} & \dots & b_n \end{vmatrix} = \cdots$$
- $$= \begin{vmatrix} 0 & \dots & 0 & -\sum_{i=1}^{n} a_{1i}b_{i1} & \dots & -\sum_{i=1}^{n} a_{1i}b_{in} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & -\sum_{i=1}^{n} a_{ni}b_{i1} & \dots & -\sum_{i=1}^{n} a_{ni}b_{in} \\ 1 & \dots & 0 & b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & b_{n1} & \dots & b_{nn} \end{vmatrix}$$
- $$= \begin{vmatrix} 0 & -AB \\ I & B \end{vmatrix}$$

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Monday, November 6, 2017

Find the Inverse of Matrix

- Gauss-Jordan Elimination
  - $(A|I) \sim (I|A^{-1})$
- Example
  - $\left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & & \\ 3 & 5 & -7 & & 1 & \\ 0 & 0 & 1 & & & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & -1 & -13 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$
  - $\to \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & -1 & -13 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & -1 & 0 & -3 & 1 & 13 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$
  - $\to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -4 \\ 0 & -1 & 0 & -3 & 1 & 13 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 2 & 22 \\ 0 & -1 & 0 & -3 & 1 & 13 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$
  - $\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 2 & 22 \\ 0 & 1 & 0 & 3 & -1 & -13 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$
  - Therefore $\begin{pmatrix} 1 & 2 & 4 \\ 3 & 5 & -7 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -5 & 2 & 22 \\ 3 & -1 & -13 \\ 0 & 0 & 1 \end{pmatrix}$

Question 1

- Recall that the determinant is a polynomial in the entries of the matrix.
- Find the coefficient of t^3 in the following polynomial

$$\begin{vmatrix} 2 & 3 & -7 & t \\ 5 & t & a & b \\ t & -1 & 0 & 55 \\ 1/2 & 3 & c & -\pi \end{vmatrix}$$

- Answer: By cofactor expansion, the coefficient is c

Question 2

- Suppose A is an orthogonal matrix, meaning A is invertible and $A^{-1} = A^T$
- What possible value could the determinant of A have?
- Answer:
  - $|A^{-1}| = |A^T|$
  - $\Rightarrow \dfrac{1}{|A|} = |A|$
  - $\Rightarrow |A| = \pm 1$

Question 3

- Let V be the vector space of all (real) polynomials of degree 2 or less.
- Using the basis $1, x, x^2$, find the matrix of the linear map $T: V \to V$ given by
- $(Tf)(x) = f(x+2)$ for all $f \in V$ and $x \in \mathbb{R}$
- Answer:

- $T(1) = 1$
- $T(x) = 2 + x$
- $T(x^2) = 4 + 4x + x^2$
- $\Rightarrow m(T) = \begin{matrix} & \begin{matrix} 1 & x & x^2 \end{matrix} \\ \begin{matrix} 1 \\ x \\ x^2 \end{matrix} & \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix}$

Question 4

- Let x, y, z, w be real numbers.
- Compute the determinant of the following matrix
- Answer:
  - $\begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & y & y^2 & y^3 \\ 1 & z & z^2 & z^3 \\ 1 & w & w^2 & w^3 \end{vmatrix} = (w - z)(w - y)(w - x)(z - y)(z - x)(y - x)$

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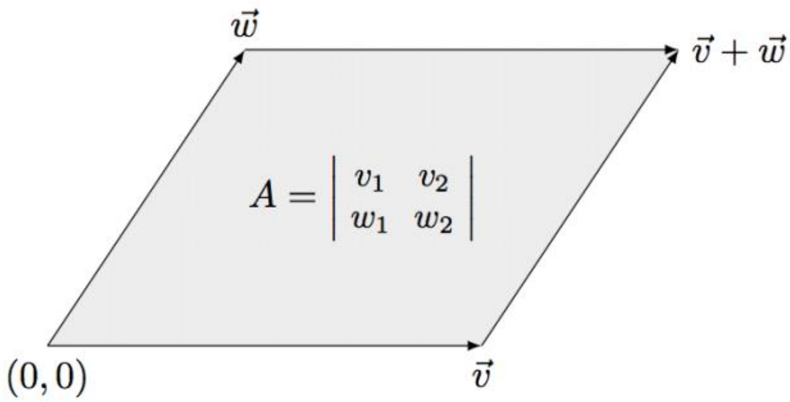
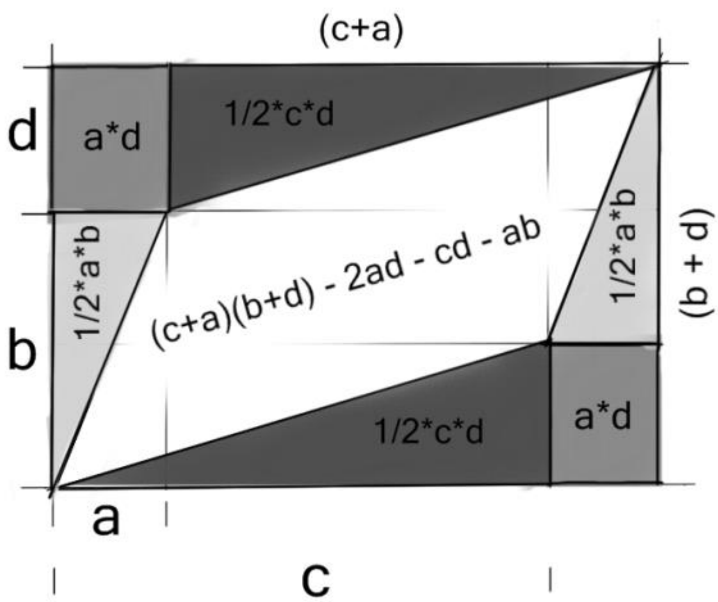
Tuesday, November 7, 2017

Determinant and Area

- $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ = area of parallelogram with sides $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

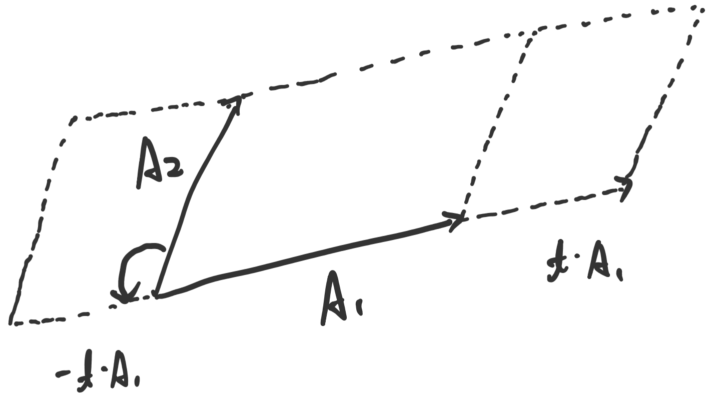


- Proof by graph



- Proof
    - Area(A_1, A_2) = signed area of parallelogram spanned by A_1, A_2
    - If $A_1 \to A_2$ is counter-clockwise = area
    - If $A_1 \to A_2$ is clockwise $= -$area
    - Then Area$(A_1, A_2) = \det(A_1, A_2)$, because
    - Alternating
        - Area$(A_1, A_2) = -$Area(A_2, A_1)
        - (by definition, same area, but different orientation)
    - Linearity(Homogeneous)
        - Area$(t \cdot A_1, A_2) = t \cdot$ Area(A_1, A_2)
        - (Easy to prove from picture)

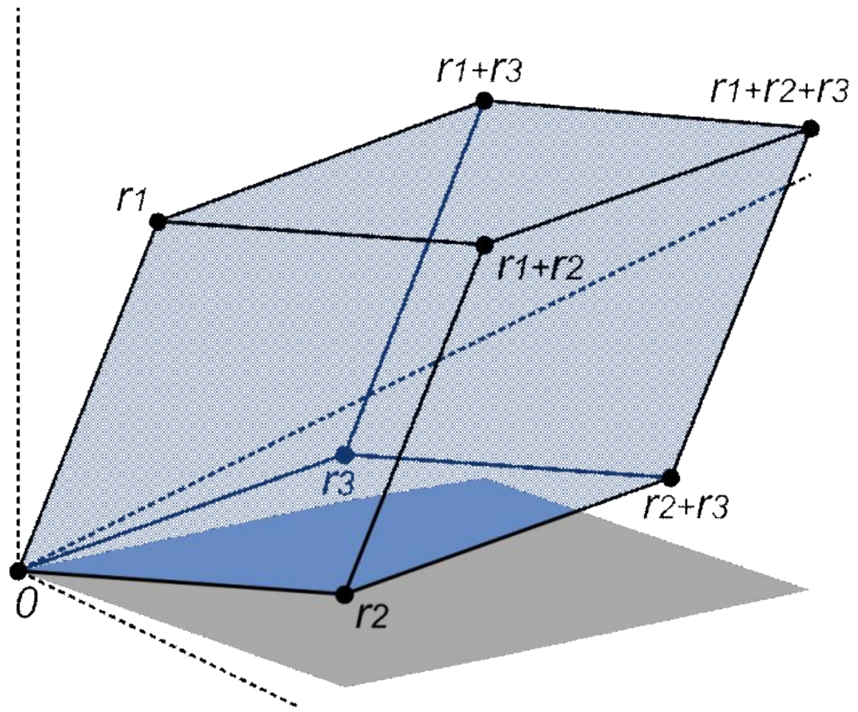
- (Easy to prove from picture)



- Linearity(Additive)
  - $\text{Area}(A+B,C) = \text{Area}(A,C) + \text{Area}(B,C)$
  - If A, C is parallel, then
    - $\text{Area}(A,C) = 0$
  - If A, C is independent , then
    - $\text{Area}(A+sC,C) = \text{Area}(A,C), \forall A, C$
  - Let $B = t \cdot A + s \cdot C$, then
    - $\text{Area}(A+B,C)$
    - $= \text{Area}(A + t \cdot A + s \cdot C, C)$
    - $= \text{Area}(A + t \cdot A, C)$
    - $= (1+t)\text{Area}(A,C)$
    - $= \text{Area}(A,C) + t \cdot \text{Area}(A,C)$
    - $= \text{Area}(A,C) + \text{Area}(t \cdot A, C)$
    - $= \text{Area}(A,C) + \text{Area}(t \cdot A + s \cdot C, C)$
    - $= \text{Area}(A,C) + \text{Area}(B,C)$
  - Therefore $\text{Area}(A+B,C) = \text{Area}(A,C) + \text{Area}(B,C)$
- Uniqueness Theorem
  - $\text{Area}(A,B)$
  - $= \det(A,B) \cdot \text{Area}(I_1, I_2)$
  - $= \det(A,B) \cdot \text{Area}(\text{unit square})$
  - $= \det(A,B)$

Determinant and Volume

- $\det(A,B,C) =$ signed volume of parallelepiped spanned by A, B, C



Inverse of a Matrix

- Setup
  - $T: \mathbb{R}^n \to \mathbb{R}^n$ linear
  - T has a matrix $m(T) = \begin{bmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nn} \end{bmatrix}$
- The following statements are equivalent
  - $N(T) = \{0\}$
  - T is injective
  - T is one-to-one
  - T is bijective
    - because $T: \mathbb{R}^n \to \mathbb{R}^n$
    - $\dim N(T) + \dim \operatorname{range}(T) = \dim \mathbb{R}^n$
    - $\Rightarrow \dim \operatorname{range}(T) = n$
    - $\Rightarrow R(T) = \mathbb{R}^n$
  - There is a map $S: \mathbb{R}^n \to \mathbb{R}^n$ with $ST = TS = I$
- Find the inverse of 2×2 matrix
  - $T = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$
  - Find T^{-1}, i.e. solve $Tx = y$
  - Note: $Tx = y \Leftrightarrow x = T^{-1}y$
  - Normal version
    - $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
    - $\begin{cases} x_1 + 3x_2 = 1 \cdot y_1 + 0 \cdot y_2 \\ 2x_2 + 5x_2 = 0 \cdot y_1 + 1 \cdot y_2 \end{cases}$

- $\Rightarrow \begin{cases} x_1 = -5y_1 + 3y_2 \\ x_2 = 2y_1 - y_2 \end{cases}$
- $\Rightarrow x = T^{-1}y$
- where $T^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$

- Shorthand
  - $[T|I]$
  - $\sim \left[\begin{matrix} 1 & 3 \\ 2 & 5 \end{matrix} \middle| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right]$
  - $\sim \left[\begin{matrix} 1 & 3 \\ 0 & -1 \end{matrix} \middle| \begin{matrix} 1 & 0 \\ -2 & 1 \end{matrix}\right]$
  - $\sim \left[\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \middle| \begin{matrix} -5 & 3 \\ -2 & 1 \end{matrix}\right]$
  - $\sim \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} -5 & 3 \\ 2 & -1 \end{matrix}\right]$
  - $\sim [I|T^{-1}]$
  - Therefore $T^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$

Minors and Cofactors

- Theorem
  - $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kn} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{k1}C_{k1} + a_{k2}C_{k2} + \cdots + a_{kn}C_{kn}$
  - C_{kl} = cofactor matrix
- Cofactor Matrix

$$C_{kl} = (-1)^{k+l} \begin{Bmatrix} (n-1) \times (n-1) \text{ determinant obtained} \\ \text{by deleting row } k \text{ and column } l \\ \text{from the original determinant} \end{Bmatrix}$$

- Example
  - $\begin{vmatrix} 1 & 7 & 2 \\ 4 & \pi & -1 \\ 3 & \ln 2 & 2 \end{vmatrix}$
  - $= 3 \times \begin{vmatrix} 7 & 2 \\ \pi & -1 \end{vmatrix} - \ln 2 \begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 7 \\ 4 & \pi \end{vmatrix}$
  - $= 3 \times (-7 - 2\pi) - \ln 2 \times (-9) + 2 \times (\pi - 28)$
  - $= -77 - 4\pi + 9 \ln 2$
- Matrix Multiplication
  - Let $P = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$
  - $P_{11} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} = \det A$
  - $P_{21} = a_{21}C_{11} + a_{21}C_{12} + \cdots + +a_{21}C_{1n} = 0$
  - Because we have two equal row

- Therefore $P = \det A \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$

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Wednesday, November 8, 2017

Effect of Row Operations on Determinants

Row Operation	Determinant
Row $A \to$ Row $A + c \cdot$ Row B	$\det M \to \det M$
Row $A \to c \cdot$ Row A	$\det M \to c \cdot \det M$
Row $A \xleftrightarrow{\text{switch}}$ Row B	$\det M \to -\det M$

Understanding of Matrix Multiplication in terms of Linear Map Composition

- Motivation
  - $V \xrightarrow{T} W \xrightarrow{S} Z$
- Setup
  - $\{e_1 \dots e_n\}$: basis of V
  - $\{f_1 \dots f_m\}$: basis of W
  - $\{g_1 \dots g_k\}$: basis of Z
  - Let $m(T) = (t_{ij})$
  - Let $m(S) = (s_{ij})$
- Claim
  - $m(S) \cdot m(T) = m(ST)$
- Proof
  - $T(e_i) = \sum_{j=1}^{m} t_{ij} f_j$
  - $S(f_j) = \sum_{k=1}^{r} s_{jk} g_k$
  - $ST(e_i) = \sum_{j=1}^{m} \sum_{k=1}^{r} t_{ij} s_{jk} g_k$
  - Which is the same as matrix multiplication

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Thursday, November 09, 2017

Expansion by Rows Theorem

- Cofactor Matrix
  - $C_{kl} = (-1)^{k+l}\left\{\begin{matrix}(n-1)\times(n-1) \text{ determinant obtained} \\ \text{by deleting row } k \text{ and column } l \\ \text{from the original determinant}\end{matrix}\right\}$
- Determinant and Cofactor Matrix
  - $\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{k1}C_{k1} + a_{k2}C_{k2} + \cdots + a_{kn}C_{kn}$
  - $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \underbrace{\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}}_{\text{adjugate matrix of } A\text{: adj}(A)} = \det A \cdot \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$
- Expansion by Rows
  - $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$
  - $\begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = x_1C_{11} + x_2C_{12} + \cdots + x_nC_{1n}$
- Calculating $A \cdot \text{adj}(A)$
  - Expanding $A \cdot \text{adj}(A)$
    - $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$
    - $= \begin{bmatrix} \sum_{k=1}^{n} a_{1k}C_{1k} & \sum_{k=1}^{n} a_{1k}C_{2k} & \cdots & \sum_{k=1}^{n} a_{1k}C_{nk} \\ \sum_{k=1}^{n} a_{2k}C_{1k} & \sum_{k=1}^{n} a_{2k}C_{2k} & \cdots & \sum_{k=1}^{n} a_{2k}C_{nk} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} a_{nk}C_{1k} & \sum_{k=1}^{n} a_{nk}C_{2k} & \cdots & \sum_{k=1}^{n} a_{nk}C_{nk} \end{bmatrix}$
  - Where
    - $\sum_{k=1}^{n} a_{1k}C_{1k} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \det A$

- $\sum_{k=1}^{n} a_{1k} C_{2k} = \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0$
- $\vdots$

- Conclusion
  - $A \cdot \text{adj}(A) = \begin{bmatrix} \det A & & & \\ & \det A & & \\ & & \ddots & \\ & & & \det A \end{bmatrix} = \det A \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$

- Theorem
  - $\det(A) \neq 0 \Leftrightarrow A$ is invertible and $A^{-1} = \frac{1}{\det A} \cdot \text{adj}(A)$
  - $\det(A) = 0 \Leftrightarrow A$ is not invertible
- Example
  - Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
  - Cofactor Matrix
    - $C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$
  - Adjugate Matrix
    - $\text{adj}(A) = C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
  - Determinant
    - $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
  - Inverse Matrix
    - $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A} \cdot \text{adj}(A) = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Cramer's Rule

- Trying to solve the following system of equations
  - $\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = y_1 \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = y_n \end{cases}$
- It can be written in matrix form
  - $Ax = y$, Where
  - $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$
  - $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$
  - $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$
- Solve x in matrix form, we get
  - $x = A^{-1}y = \frac{1}{\det A} \cdot \text{adj}(A) \cdot y$

- $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$
- In particular
  - $x_1 = \frac{1}{\det A}(C_{11}y_1 + C_{21}y_2 + \cdots + C_{n1}y_n) = \dfrac{\begin{vmatrix} y_1 & a_{12} & \cdots & a_{1n} \\ y_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}}$
- In general
  - $x_k = \frac{1}{\det A}(C_{11}y_1 + C_{21}y_2 + \cdots + C_{n1}y_n) = \dfrac{\begin{vmatrix} a_{11} & \cdots & y_1 & \cdots & a_{1n} \\ a_{21} & \cdots & y_2 & \cdots & a_{2n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{n1} & \cdots & y_n & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}}$
  - Where y_i is at the k^{th} column

Linear Independence and Determinant

- Theorem
  - Let $v_1, \dots, v_n \in \mathbb{R}^n$ be vectors with
  - $v_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix}, \dots, v_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{bmatrix}$
  - Then $\{v_1, \dots, v_n\}$ is independent $\Leftrightarrow \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \neq 0$
- Example
  - Are $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ \alpha \\ \beta \end{bmatrix}$ dependent?
  - If $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & \alpha \\ 4 & 4 & \beta \end{bmatrix} = 0$, then yes
- Proof
  - $c_1 v_1 + \cdots + c_n v_n$
  - $= c_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} + \cdots + c_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{bmatrix}$
  - $= \begin{bmatrix} a_{11}c_1 + a_{12}c_2 + \cdots + a_{1n}c_n \\ \vdots \\ a_{n1}c_1 + a_{n2}c_2 + \cdots + a_{nn}c_n \end{bmatrix}$
  - $= \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}_{c}$
  - Prove: $\det A \neq 0 \Rightarrow \{v_1, \dots, v_n\}$ is linearly independent

- Suppose $\det A \neq 0$, then A^{-1} exists
- If $c_1 v_1 + \cdots + c_n v_n = 0$, then $Ac = 0$
- And $c = A^{-1}Ac = A^{-1}(0) = 0$
- So $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$
- i.e. $c_1 = c_2 = \cdots = c_n = 0$
- Therefore $\{v_1, \dots, v_n\}$ is linearly independent

○ Prove: $\det A = 0 \Rightarrow \{v_1, \dots, v_n\}$ is linearly dependent

- Suppose $\det A = 0$
- Then A is not invertible
- Since A is a square matrix this means A is not injective
- Therefore $N(A) \neq \{0\}$
- i.e. There exists a vector $c \neq 0$ with $Ac = 0$
- Since $Ac = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$
- We can find that there are $c_1, \dots c_n$,
- at least one of which is non-zero with $c_1 v_1 + \cdots + c_n v_n = 0$
- Therefore $\{v_1, \dots, v_n\}$ is linearly dependent

11/13

Monday, November 13, 2017

Question 1

- $A = \begin{pmatrix} 1 & 1 & a \\ -1 & 1 & b \\ 0 & 2 & c \end{pmatrix}$
- For which $a, b, c \in \mathbb{R}$ is A invertible?
- When A is invertible, find A^{-1}
- Answer:
  - $\det A = -2a - 2b + 2c$
  - $\text{cof A} = \begin{pmatrix} c-2b & c & -2 \\ 2a-c & c & -2 \\ b-a & -a-b & 2 \end{pmatrix}$
  - $\text{adj A} = (\text{cof A})^{\mathrm{T}} = \begin{pmatrix} c-2b & 2a-c & b-a \\ c & c & -a-b \\ -2 & -2 & 2 \end{pmatrix}$
  - $A^{-1} = \dfrac{1}{-2a-2b+c}\begin{pmatrix} c-2b & 2a-c & b-a \\ c & c & -a-b \\ -2 & -2 & 2 \end{pmatrix}$
  - Where $a + b \neq c$

Question 2

- Let A be square matrix such that $A^k = 0$ for some k
- Prove or find a counterexample : $I - A$ is invertible
- Answer:
  - $I = I - A^k = (I - A)\left(I + A + A^2 + \cdots A^{k-1}\right)$
  - Therefore $I - A$ is invertible
- Note:
  - A is called Nilpotent matrix

11/14

Tuesday, November 14, 2017

Eigenvalues and Eigenvectors

- Definition
  - If $T: V \to V$ is linear and V is a vector space
  - Then $v \in V$ is an eignevector of T with eigenvalue λ if
    - $v \neq 0$
    - $Tv = \lambda v$
- Example
  - Suppose you have two eigenvectors
    - $v, w \in V$ with $Tv = \lambda v, Tw = \mu w$
  - Then
    - $T(2v + 3w) = 2Tv + 3Tw = 2\lambda v + 3\mu w$
  - Find a solution to $Tx = v + w$
    - Try $x = av + bw$
    - Then $Tx = T(av + bw)$
    - $= \lambda av + \mu bw$
    - $= v + w$
    - $\overset{?}{\Rightarrow} \begin{cases} \lambda a = 1 \\ \mu b = 1 \end{cases} \Rightarrow \begin{cases} a = \dfrac{1}{\lambda} \\ b = \dfrac{1}{\mu} \end{cases}$ (if $\lambda, \mu \neq 0$)
    - Therefore $x = \dfrac{1}{\lambda} v + \dfrac{1}{\mu} w$
  - Compute $T^{2017}(2v + 3w)$
    - $T^{2017}(2v + 3w)$
    - $= T^{2016}(2\lambda v + 3\mu w)$
    - $= T^{2015}(2\lambda^2 v + 3\mu^2 w)$
    - $\vdots$
    - $= 2\lambda^{2017} v + 3\mu^{2017} w$
- Fibonacci Number
  - $f_n = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ f_{n-1} + f_{n-2} & n \geq 2 \end{cases}$
  - For example
    - $f_0 = 0$
    - $f_1 = 1$
    - $f_2 = 1$
    - $f_3 = 2$

- $f_4 = 3$
- $\vdots$

- It could be viewed as a sequence of vectors
  - $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}3\\2\end{bmatrix} \ldots$
- Consider
  - $x_n = \begin{bmatrix}f_n\\f_{n-1}\end{bmatrix}$
  - $x_{n+1} = \begin{bmatrix}f_{n+1}\\f_n\end{bmatrix} = \begin{bmatrix}f_n + f_{n+1}\\f_n\end{bmatrix} = \underbrace{\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}}_{T}\underbrace{\begin{bmatrix}f_n\\f_{n-1}\end{bmatrix}}_{x_n}$
- Try to compute
  - $x_{2017} = \begin{bmatrix}f_{2017}\\f_{2016}\end{bmatrix} = T\begin{bmatrix}f_{2016}\\f_{2015}\end{bmatrix} = \cdots = T^{2016}\begin{bmatrix}1\\0\end{bmatrix}$
  - If we had two eigenvectors/eigenvalues for T
  - And $\begin{bmatrix}1\\0\end{bmatrix} = av + bw$
  - Then $\begin{bmatrix}f_{2017}\\f_{2016}\end{bmatrix} = \lambda^{2016}av + \mu^{2016}bw$

- Eigenvector Equation
  - By definition, if $T: V \to V$ is linear and V is a vector space
  - Then $v \in V$ is an eignevector of T with eigenvalue if
    - $v \neq 0$, and $Tv = \lambda v$
    - $\Rightarrow Tv = \lambda Iv$
    - $\Rightarrow Tv - \lambda Iv = 0$
    - $\Rightarrow (T - \lambda I)v = 0$
    - $\Leftrightarrow v \in \text{Null}(T - \lambda I)$
  - Therefore
    - v is an eigenvector with eigenvalue λ
    - $\Rightarrow 0 \neq v \in \text{Null}(T - \lambda I)$
    - $\Rightarrow T - \lambda I$ is not injective
- Theorem
  - If $T\colon\ \mathbb{R}^n \to \mathbb{R}^n$ is given by matrix multipication
  - Then λ is an eigenvalue of T if and only if
  - $\det(T - \lambda I) = 0$
- Proof
  - $V = \mathbb{R}^n$ or $\mathbb{C}^n$
  - $Tx = \begin{bmatrix}t_{11} & \cdots & t_{1n}\\ \vdots & \ddots & \vdots\\ t_{n1} & \cdots & t_{nn}\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$
  - $T - \lambda I = \begin{bmatrix}t_{11} - \lambda & t_{12} & \cdots & t_{1n}\\ t_{21} & t_{22} - \lambda & \ldots & t_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ t_{n1} & t_{n2} & \cdots & t_{nn} - \lambda\end{bmatrix}$

- Fibonacci Example
  - $T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$
  - $\det(T - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 \stackrel{?}{=} 0$

- Solving for eigenvalue and eigenvector
  - For $T: \mathbb{R}^n \to \mathbb{R}^n$ (or $\mathbb{C}^n \to \mathbb{C}$)
  - $\det(T - \lambda I)$ is called the characteristic polynimal of T
    - $\det(T - \lambda I)$
    - $= \begin{vmatrix} t_{11} - \lambda & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} - \lambda & \dots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n1} & t_{n2} & \cdots & t_{nn} - \lambda \end{vmatrix}$
    - $= (-\lambda)^n + c_1(-\lambda)^{n-1} + \cdots + c_{n-1}(-\lambda) + c_n$
    - Where $c_1 = tr(T),\ c_n = \det T$
  - By Fundamental Theorem of Algebra
    - $\det(T - \lambda I)$
    - $= (-\lambda)^n + c_1(-\lambda)^{n-1} + \cdots + c_{n-1}(-\lambda) + c_n$
    - $= (-\lambda)^n(\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$
    - $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$ is called the eigentvalue of T
  - Given eigenvalues $\lambda_1, \dots, \lambda_n$
    - We can find eigenvectors $N_1, \dots, N_n$ by
    - $N_1 \in N(T - \lambda_1 I)$
    - $N_2 \in N(T - \lambda_2 I)$
    - $\vdots$
    - $N_n \in N(T - \lambda_n I)$

- Theorem
  - $T: V \to V$
  - $v_1, \dots, v_k \in V$ are eigenvectors
  - with distinct eigenvalues $\lambda_1, \dots, \lambda_k$
  - then $\{v_1, \dots, v_k\}$ is linearly indelendent

- Proof
  - By induction on k
  - When $k = 1$
    - Given $v_1 \in V, v_1 \neq 0, Tv_1 = \lambda_1 v_2$
    - Then $\{v_1\}$ is independent because $v_1 \neq 0$
  - When $k > 1$
    - Assume Theorem true for $k - 1$
    - Suppose $Tv_1 = \lambda_1 v_1, \dots, Tv_k = \lambda_k v_k$
    - $\lambda_i \neq \lambda_j$ for all $i \neq j$, and all $v_i \neq 0$
    - Suppose $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$

- $\Rightarrow \begin{cases} \lambda_k c_1 v_1 + \lambda_k c_2 v_2 + \cdots + \lambda_k c_k v_k = 0 \\ \lambda_1 c_1 v_1 + \lambda_1 c_2 v_2 + \cdots + \lambda_1 c_k v_k = 0 \end{cases}$
- $\Rightarrow (\lambda_k - \lambda_1) c_1 v_1 + \cdots + (\lambda_k - \lambda_{k-1}) c_{k-1} v_{k-1} = 0$
- Since Theorem is true for $k-1$
- $\Rightarrow \{v_1, \dots, v_{k-1}\}$ is linearly independent
- $\Rightarrow \begin{cases} \underbrace{(\lambda_k - \lambda_1)}_{\neq 0} c_1 = 0 \\ \vdots \\ \underbrace{(\lambda_k - \lambda_{k-1})}_{\neq 0} c_{k-1} = 0 \end{cases}$
- $\Rightarrow c_1 = c_2 = \cdots = c_{k-1} = 0$
- Therefore $c_k v_k = 0$
- Since $v_k \neq 0$, we find $c_k = 0$
- $\Rightarrow \{v_1, \dots, v_k\}$ is linearly independet

11/15

Wednesday, November 15, 2017

Theorem

- Statement
  - If $\dim V = \dim W < \infty$, then for linear map $T: V \to W$
  - injective $\Leftrightarrow$ surjective $\Leftrightarrow$ bijective
- Proof
  - By Rank-Nullity Theorem
    - $\dim W = \dim V = \dim N(T) + \dim \text{Range}(T)$
  - If T is injective
    - $\Rightarrow \dim N(T) = 0$
    - $\Rightarrow \dim W = \dim \text{Range}(T)$
    - $\Rightarrow T$ is surjective
    - $\Rightarrow T$ is bijective
  - If T is not injective
    - $\Rightarrow \dim N(T) > 0$
    - $\Rightarrow \dim W \neq \dim \text{Range}(T)$
    - $\Rightarrow T$ is not surjective
    - $\Rightarrow T$ is not bijective

Left Inverse and Right Inverse

- If both left inverse and right inverse exists
- Then they are the same
- Suppose
  - $f: V \to W$
  - $g, h: W \to V$
  - $gf = id_V$ (i.e. g is the left inverse of T)
  - $fh = id_W$ (i.e. h is the right inverse of T)
- Then
  - $g = g(fh) = (gf)h = h$

Injective and Null Space

- Proof: T injective $\Rightarrow N(T) = \{0\}$
  - If T is injective
  - then the only one element mapped to 0 is 0 itself.
  - Therefore $N(T) = \{0\}$
- Proof: $N(T) = \{0\} \Rightarrow T$ injective
  - If $T(x) = T(y)$, then
  - $T(x) - T(y) = T(x - y) = 0$

- So $x - y \in N(T)$
- $\Rightarrow x = y$
- Therefore T is injective

11/20

Monday, November 20, 2017

Eigenvalues and Eigenvectors

- Definition
  - $T: V \to V$ linear, for $\begin{cases} x \in V \\ \lambda \in \mathbb{C} \end{cases}, (x \neq 0)$
  - We say x is an eigentvector for T with eigenvalue λ if $Tx = \lambda x$
- Note
  - $Tx = \lambda x$
  - $\Rightarrow Tx - \lambda x = 0$
  - $\Rightarrow (T - \lambda I)x = 0$
  - $\Rightarrow x \in N(T - \lambda I)$

Find all eigenvalues and eigenvectors

- $T = I$
  - $Tx = 1x, \qquad \forall x \in V$
  - Eigenvalue $= 1$ with eigenvectors of all elements in V
- $T = 0$
  - $Tx = 0x, \qquad \forall x \in V$
  - Eigenvalue $= 0$ with eigenvectors of all elements in V
- $T = \begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{bmatrix}, \qquad (c_i \neq c_j \text{ for } i = j)$
  - $\begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{bmatrix} e_i = c_i e_i$
  - Eigenvalue $= c_i$ with eigenvector of $te_i, (t \in \mathbb{R}, t \neq 0)$
- $T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$
  - $\det(T - \lambda I) = 0$
    - $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$
    - $(\lambda - 3)(\lambda + 1) = 0$
  - $\lambda = 3$
    - $\begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = y$
    - Eigenvector: $\begin{bmatrix} t \\ t \end{bmatrix} (t \in \mathbb{R}, t \neq 0)$
  - $\lambda = -1$
    - $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x + y = 0$
    - Eigenvector: $t \begin{bmatrix} 1 \\ -1 \end{bmatrix} (t \in \mathbb{R}, t \neq 0)$

- $T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
    - $\det(T - \lambda I) = 0$
        - $\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$
        - $\lambda^2 + 1 = 0$
    - $\lambda = i$
        - $\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow y = -ix$
        - Eigenvector: $t \begin{bmatrix} 1 \\ -i \end{bmatrix} (t \in \mathbb{C}, t \neq 0)$
    - $\lambda = -i$
        - $\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow y = ix$
        - Eigenvector: $t \begin{bmatrix} 1 \\ i \end{bmatrix} (t \in \mathbb{C}, t \neq 0)$

Multiplicity of Eigenvalues

- $T = \begin{bmatrix} 3 & & \\ & 3 & \\ & & 4 \end{bmatrix}$
    - Eigenvalues: $\lambda = 3$ or $\lambda = 4$
    - $\dim N(T - \lambda I) = \begin{cases} 2 & \lambda = 3 \\ 1 & \lambda = 4 \\ 0 & \text{otherwise} \end{cases}$

11/21

Tuesday, November 21, 2017

Eigenvalues and Eigenvectors

- Definition
  - If $T: V \to V$ is linear and V is a vector space
  - Then $v \in V$ is an eignevector of T with eigenvalue λ if
    - $v \neq 0$
    - $Tv = \lambda v$
- Theorem
  - Linear transformation $\mathcal{T}: \mathbb{R}^n \to \mathbb{R}^n$ (or $\mathbb{C}^n \to \mathbb{C}^n$)
  - $\text{matrix}(\mathcal{T}) = T = \begin{bmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{n1} & \cdots & t_{nn} \end{bmatrix}$
  - Then λ is an eigenvalue of $\mathcal{T}$ if
  - $\det(T - \lambda I) = 0$
- Characteristic Polynomial
  - $\det(T - \lambda I)$ is the called characteristic polynomial of T
  - $f(\lambda) = \det(T - \lambda I) = \begin{vmatrix} t_{11} - \lambda & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} - \lambda & \dots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n1} & t_{n2} & \cdots & t_{nn} - \lambda \end{vmatrix}$
  - $= (-\lambda)^n + c_1(-\lambda)^{n-1} + \cdots + c_{n-1}(-\lambda) + c_n$
- How to Find Eigenvalues
  - Solve $\det(T - \lambda I) = 0$
  - Get roots $\lambda_1, \dots, \lambda_n$ (possibly repeated)
- How to Find Eigenvectors
  - Solve $(T - \lambda I)v = 0$
  - For $\lambda = \lambda_1, \lambda = \lambda_2, \dots, \lambda = \lambda_n$
  - $(T - \lambda I)v = 0$ is n equations with n unknowns
  - Typically $v = 0$ is the only solution for some $\lambda = \lambda_i$
  - Then $\det(T - \lambda I) = 0$, and there is a solution $v \neq 0$
- Coefficients of Characteristic Polynomial
  - By definition
    - $f(\lambda) = (-\lambda)^n + c_1(-\lambda)^{n-1} + \cdots + c_{n-1}(-\lambda) + c_n$
  - By Fundamental Theorem of Algebra
    - $f(\lambda) = a(\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$
  - Comparing the coefficient of $(-\lambda)^n$, we get
    - $a = 1$
  - Setting $\lambda = 0$ to both polynomials we get

        - $c_n = \det T = \lambda_1 \lambda_2 \dots \lambda_n$
    - By Vieta's Formula
        - $c_1 = \lambda_1 + \lambda_2 + \cdots + \lambda_n$
        - $c_2 = \sum_{1 \le i < j \le n} \lambda_i \lambda_j$
    - Expand the first row of determinant to find c_1
        - $\begin{vmatrix} t_{11} - \lambda & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} - \lambda & \dots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n1} & t_{n2} & \cdots & t_{nn} - \lambda \end{vmatrix}$
        - $= \begin{vmatrix} -\lambda & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} - \lambda & \dots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n1} & t_{n2} & \cdots & t_{nn} - \lambda \end{vmatrix} + \begin{vmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} - \lambda & \dots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n1} & t_{n2} & \cdots & t_{nn} - \lambda \end{vmatrix}$
        - $= (-\lambda) \begin{vmatrix} t_{22} - \lambda & \dots & t_{2n} \\ \vdots & \ddots & \vdots \\ t_{n2} & \cdots & t_{nn} - \lambda \end{vmatrix} + t_{11} \underbrace{\begin{vmatrix} t_{22} - \lambda & \dots & t_{2n} \\ \vdots & \ddots & \vdots \\ t_{n2} & \cdots & t_{nn} - \lambda \end{vmatrix}}_{\text{terms with } (-\lambda)^{n-1}} + \cdots$
        - $= (-\lambda) \underbrace{\begin{vmatrix} t_{22} - \lambda & \dots & t_{2n} \\ \vdots & \ddots & \vdots \\ t_{n2} & \cdots & t_{nn} - \lambda \end{vmatrix}}_{\text{looking for terms with } (-\lambda)^{n-2}} + t_{11}(-\lambda)^{n-1} + \cdots$
        - Repeat this procedure, we get
        - $c_1 = t_{11} + t_{22} + \cdots + t_{nn}$
    - Note:
        - $c_1 = \lambda_1 + \lambda_2 + \cdots + \lambda_n = t_{11} + t_{22} + \cdots + t_{nn}$
- Theorem
    - $\prod_{i=1}^{n} \lambda_i = \det T$
    - $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} t_{ii}$
- Trace of Matrix
    - Matrix: $T = \begin{bmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{n1} & \cdots & t_{nn} \end{bmatrix}$
    - Eigenvalues: $\lambda_1, \dots, \lambda_n$
    - Characteristic polynomial: $f(\lambda)$
    - The sum of the roots of $f(\lambda)$ is called the trace of T, denoted as $\operatorname{tr}(T)$
    - $\operatorname{tr}(T) = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} t_{ii}$
- Theorem
    - If $v_1, \dots, v_k$ are eigenvectors of T with eigenvalues $\lambda_1, \dots, \lambda_k$
    - And if $\lambda_i \neq \lambda_j$ for $i \neq j$
    - Then $\{v_1, \dots, v_k\}$ is linearly independent

- Theorem:
  - If T is a $n \times n$ matrix and all eigenvalues are different
  - Then $\{v_1, \dots v_n\}$ is a basis for $\mathbb{R}^n$ (or $\mathbb{C}^n$)
- Diagonalization
  - T is the linear transformation with eigenvectors $v_1, \dots, v_n$
  - Consider $V: \mathbb{R}^n \to \mathbb{R}^n$
    - $V\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \stackrel{\text{def}}{=} x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$
    - $Ve_k = 0v_1 + \cdots + 1v_k + \cdots + 0v_n = v_k$
  - Matrix of V
    - Let $v_1 = \begin{bmatrix} v_{11} \\ \vdots \\ v_{n1} \end{bmatrix}, v_1 = \begin{bmatrix} v_{12} \\ \vdots \\ v_{n2} \end{bmatrix}, \dots, v_n = \begin{bmatrix} v_{1n} \\ \vdots \\ v_{nn} \end{bmatrix}$
    - $V = \begin{bmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nn} \end{bmatrix}$
  - V is invertible
    - Because if $x = x_1 e_1 + \cdots + x_n e_n \in N(V)$
    - Then $Vx = x_1 v_1 + \cdots + x_n v_n = 0$
    - $\{v_1, \dots, v_n\}$ is linearly independent
    - $\Rightarrow x_1 = x_2 = \cdots = x_n = 0$
    - $\Rightarrow N(V) = \{0\}$
  - Let $\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$
    - $\Lambda: \mathbb{R}^n \to \mathbb{R}^n, \Lambda e^k = \lambda e^k$
    - Let $x = x_1 e_1 + \cdots + x_n e_n$
    - $V\Lambda x$
      - $= V\big(\Lambda(x_1 e_1 + \cdots + x_n e_n)\big)$
      - $= V(x_1 \Lambda e_1 + \cdots + x_n \Lambda e_n)$
      - $= V(x_1 \lambda_1 e_1 + \cdots + x_n \lambda_n e_n)$
      - $= x_1 \lambda_1 V e_1 + \cdots + x_n \lambda_n V e_n$
      - $= x_1 \lambda_1 v_1 + \cdots + x_n \lambda_n v_n$
    - TVx
      - $= T\big(V(x_1 e_1 + \cdots + x_n e_n)\big)$
      - $= T(x_1 v_1 + \cdots + x_n v_n)$
      - $= xTv_1 + \cdots + xTv_n$
      - $= x_1 \lambda_1 v_1 + \cdots + x_n \lambda_n v_n$
    - Therefore $TV = V\Lambda$
    - Multiply V^{-1} on the left, we have
      - $V^{-1}TV = V^{-1}V\Lambda = \Lambda$
    - Multiply V^{-1} on the right, we have

- $T = TVV^{-1} = V\Lambda V^{-1}$

- Application
  - If you knew Λ, V, V^{-1}, then
  - $T^m = (V\Lambda V^{-1})^m = V\Lambda V^{-1} \cdot V\Lambda V^{-1} \cdots V\Lambda V^{-1} = V\Lambda^m V^{-1}$
  - Λ^m is easy to calculate: $\Lambda^m = \begin{bmatrix} \lambda_1^m & & \\ & \ddots & \\ & & \lambda_n^m \end{bmatrix}$

11/22

Wednesday, November 22, 2017

Theorem

- V has a basis $v_1, \ldots, v_n$, and another basis $w_1, \ldots, w_n$
- Let T be a linear transformation $V \to V$
- Define the following matrices
    - $A \coloneqq \text{matrix}(T, v_i)$
    - $B \coloneqq \text{matrix}(T, w_i)$
    - $C \coloneqq \forall i \in \{1, \ldots, n\}, C(w_i) = v_i$
- Then $B = C^{-1}AC$

Question

- Given
    - $T: \mathbb{R}^3 \to \mathbb{R}^3$
    - $f(T) = (2 - \lambda)^2(3 - \lambda)$
    - $\dim\big(Null(T - 2I)\big) = 1$
- Find T
    - $T = \begin{bmatrix} 2 & 1 & 0 \\ * & 2 & 0 \\ 0 & * & 3 \end{bmatrix}$
    - For $\lambda = 2$
    - $Tv = 2v$
    - $\Rightarrow v = k\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

11/27

Monday, November 27, 2017

Question 1

- Question
  - Let $\theta \in \mathbb{R}$.
  - Find all eigenvalues and eigenvectors of the following matrix
  - $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
- Answer
  - $|A - \lambda I| = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = (\cos\theta - \lambda)^2 + \sin^2\theta = 0$
  - $\Rightarrow \lambda^2 - (2\cos\theta)\lambda + 1 = 0$
  - $\Rightarrow \lambda = \cos\theta \pm i\sin\theta$
  - When $\lambda_1 = \cos\theta - i\sin\theta$
    - $\begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$
    - $\begin{cases} i\sin\theta\, x_1 - \sin\theta\, x_2 = 0 \\ \sin\theta\, x_1 + i\sin\theta\, x_2 = 0 \end{cases} \Rightarrow ix_1 = x_2$
    - $\Rightarrow v_1 = t(1, i), \qquad t \in \mathbb{C}$
  - When $\lambda_2 = \cos\theta + i\sin\theta$
    - $\begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$
    - $\begin{cases} -i\sin\theta\, x_1 - \sin\theta\, x_2 = 0 \\ \sin\theta\, x_1 - i\sin\theta\, x_2 = 0 \end{cases} \Rightarrow -ix_1 = x_2$
    - $\Rightarrow v_1 = t(1, -i), \qquad t \in \mathbb{C}$

Question 2

- Question
  - Let V be a vector space and let $T: V \to V$ be a linear map
  - Suppose $x \in V$ is an eigenvector for T with eigenvalue λ.
  - Prove that, for each polynomial,
  - the linear map $P(T)$ has eigenvector x with eigenvalue $P(\lambda)$
- Answer
  - Let $P(\lambda) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$
  - $\big(P(T)\big)(x)$
  - $= (c_nT^n + c_{n-1}T^{n-1} + \cdots + c_1T + c_0)(x)$
  - $= c_nT^n(x) + c_{n-1}T^{n-1}(x) + \cdots + c_1T(x) + c_0x$
  - $= c_n\lambda^n x + c_{n-1}\lambda^{n-1}x + \cdots + c_1\lambda x + c_0x$
  - $= (c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0)x$
  - $= \big(P(\lambda)\big)x$

Question 3

- Given
  - Let V be a vector space and let $T: V \to V$ be a linear map
  - Let c be a scalar.
  - Suppose T^2 has an eigenvalue c^2
- Prove
  - T has either c or $-c$ as an eigenvalue
- Proof
  - $\exists x \in V, \neq 0$
  - $(T^2 - c^2 I)x = 0$
  - $(T + cI)[(T - cI)x] = 0$
  - When $(T - cI)x \neq 0$
    - $(T - cI)x$ is a eigenvector for T with eigenvalue of $-c$
  - When $(T - cI)x = 0$
    - x is a eigenvector for T with eigenvalue of c

Question 4

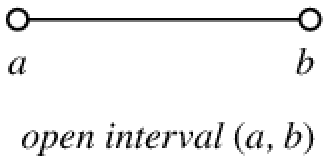
- Given
  - Let V be a vector space and let $T: V \to V$ be a linear map
  - Suppose $x, y \in V$ are eigenvectors of T with eigenvalues λ and μ.
- Prove
  - If $ax + by$ $(a, b \in \mathbb{R})$ is an eigenvector of T,
  - then $a = 0$ or $b = 0$ or $\lambda = \mu$
- (To be continued)

11/28

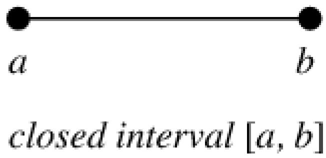
Tuesday, November 28, 2017

Open Balls and Open Sets

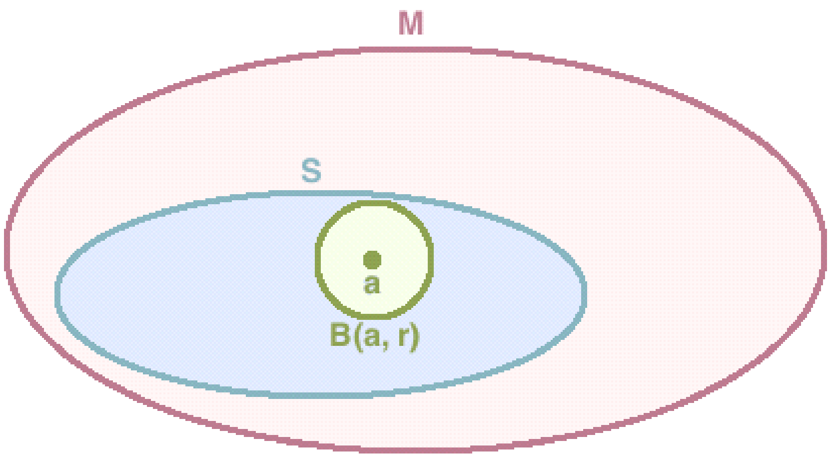
- Open Interval



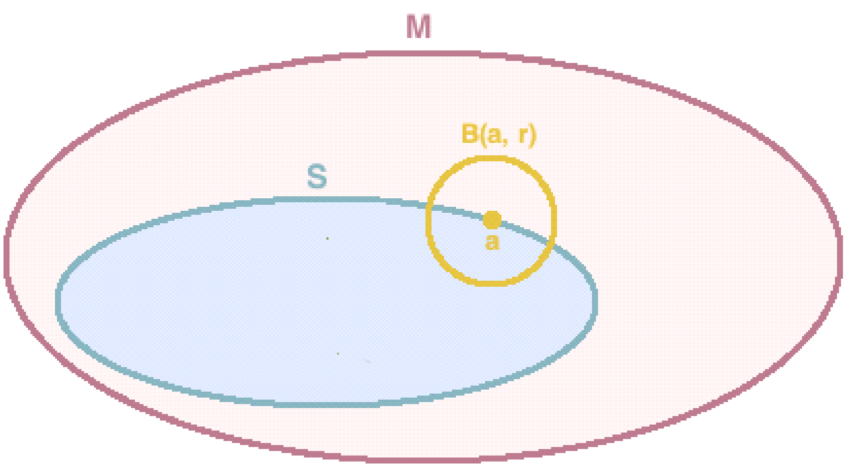
- Closed Interval



- Interior Point
    - $E \subseteq \mathbb{R}^n$ is a subset
    - $p \in E$ is an interior point if there is an $r > 0$
    - such that $B_r(p) \subseteq E$
    - where $B_r(p)$ is the open disc of radius centered at p
    - $B_r(p) = \{x \in \mathbb{R}^n | \|x - p\| < r\}$

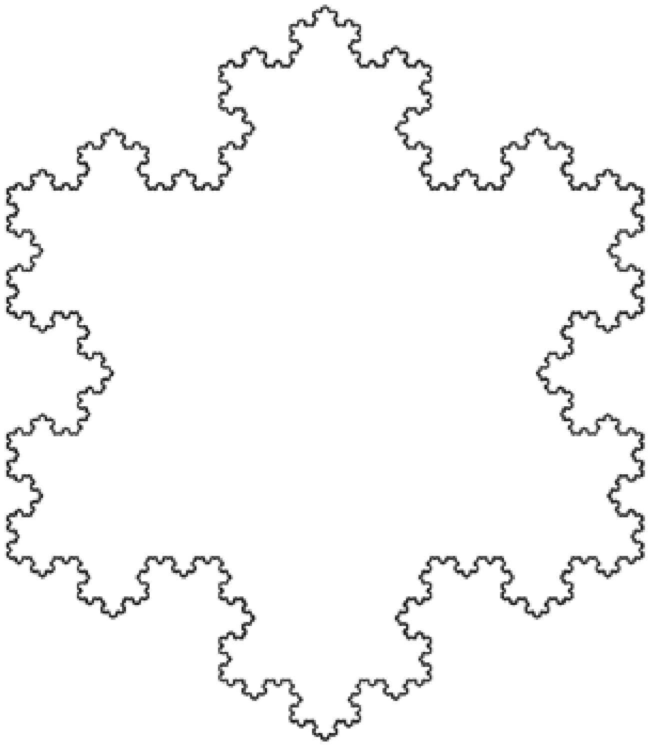


The point a is an interior point of S.

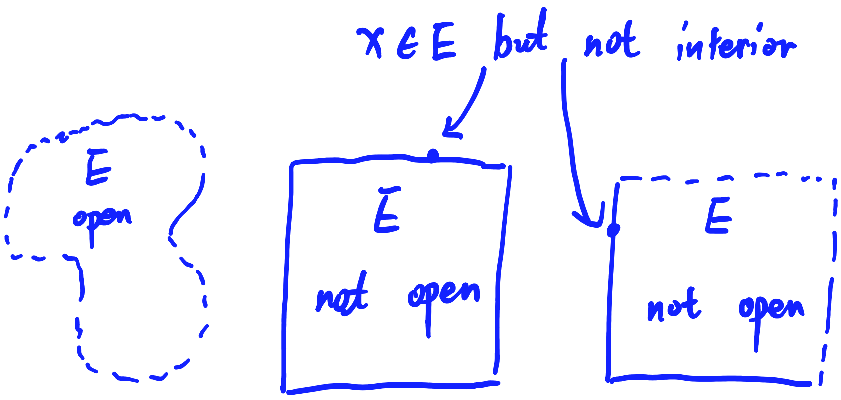


The point a is a boundary point of S.

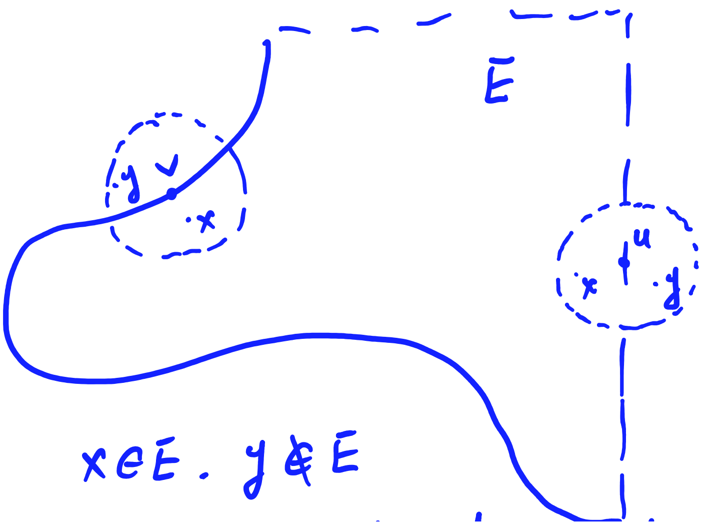
- Koch's Snowflake



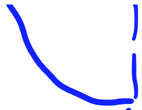
- Open Sets
  - $E \subseteq \mathbb{R}^n$ is open if all $x \in E$ are interior points in E
- Example



- Boundary Point
  - A point $p \in \mathbb{R}^n$ is a boundary point for E if for every $r > 0$
  - $B_r(p)$ contains x, y with $x \in E$ and $y \notin E$



$x \in E$. $y \notin E$

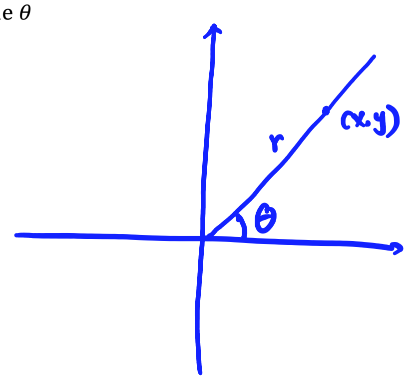


u.V one both boundary point

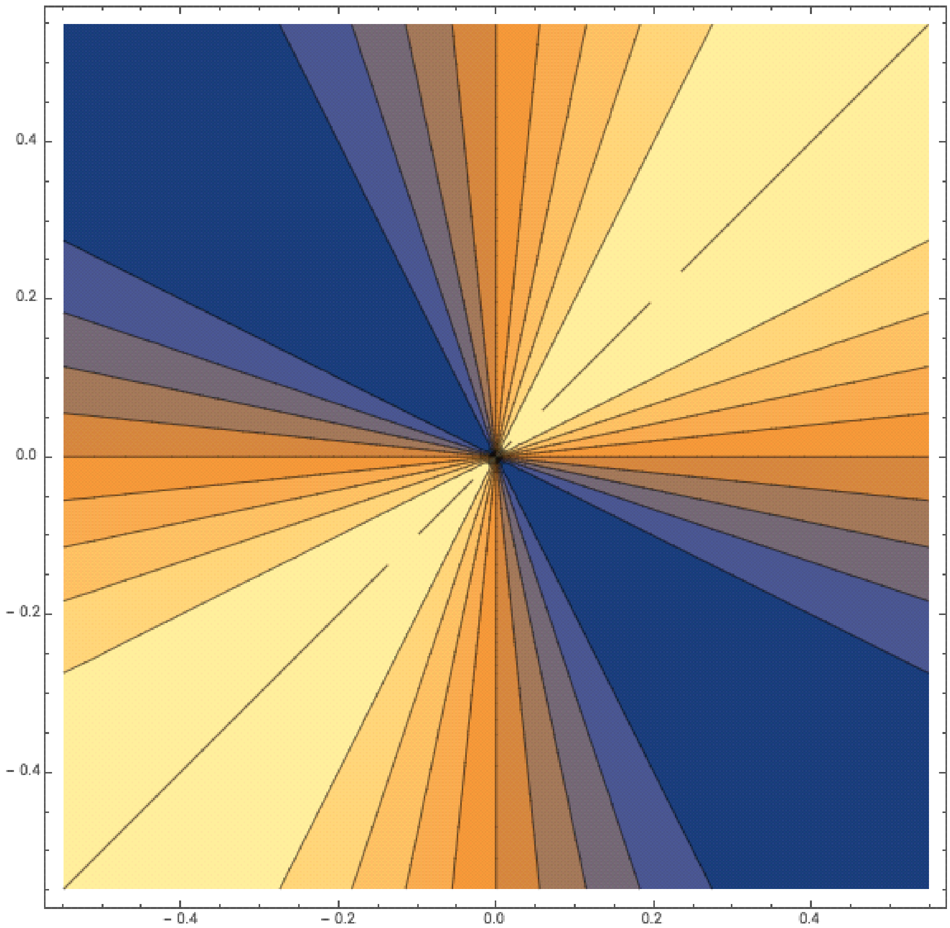
Limits and Continuity

- Limits
  - $\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{\|x-a\| \to 0} \|f(x) - L\| = 0$
  - If $x \to a$, then $f(x) \to L$
- Properties
  - If $f(x) \to L \in \mathbb{R}^m, g(x) \to M \in \mathbb{R}^m$, when $x \to a$, then
  - $f(x) \pm g(x) \to L \pm M$
  - $f(x) \cdot g(x) \to L \cdot M$
  - $\|f(x)\| \to \|L\|$
  - $\dfrac{f(x)}{g(x)} \to \dfrac{L}{M}$
  - (only when $n = 1, f(x), g(x) \in \mathbb{R}^n$)
- Graph
  - Graph of $f = \{(x, y, z) | z = f(x, y)\}$
- Continuity
  - $f: \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $a \in \mathbb{R}^n$
  - if $\lim_{x \to a} f(x) = f(a)$
- Continuous Function Example
  - $f(x_1, \dots, x_n) = x_k$
  - $f: \mathbb{R}^n \to \mathbb{R}$
- Properties
  - If f, g is continuous
  - Then $f \pm g, fg, \dfrac{f}{g} (g(a) \neq 0)$ are continuous
- Example
  - $f: \mathbb{R}^2 \to \mathbb{R}$
  - $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & (x, y) \neq (0,0) \\ 0 & x = y = 0 \end{cases}$
  - f is continuous at all point except $(0,0)$
  - Let $(x, y) \to (0,0)$ along a straight line with angle θ
  - $x = r\cos\theta, \quad y = r\sin\theta$
  - $f(x, y) = \dfrac{xy}{x^2 + y^2} = \dfrac{r^2 \sin\theta \cos\theta}{r^2 \cos^2\theta + r^2 \sin\theta} = \cos\theta \sin\theta$
  - Note that $f(x, y)$ does not depend on r

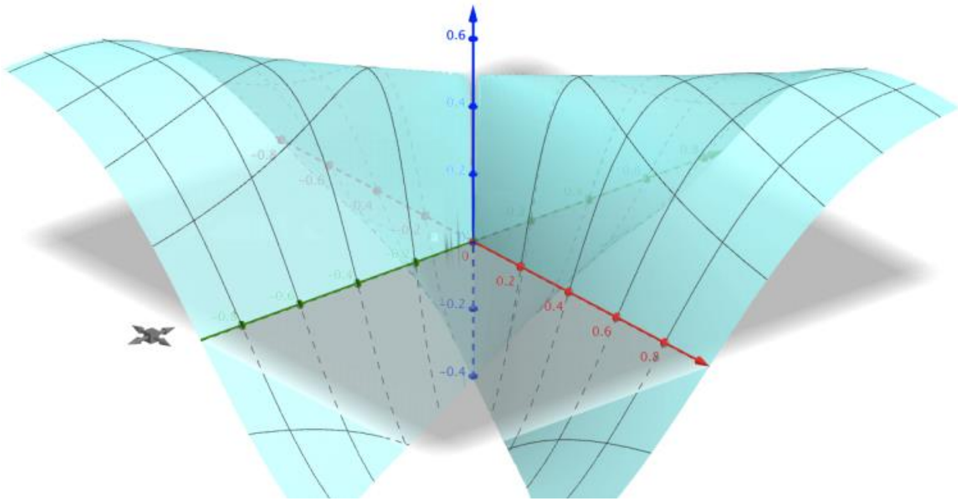
- $\lim_{\substack{(x,y)\to(0,0) \\ \text{along line} \\ \text{with angle } \theta}} f(x,y) = \sin\theta\cos\theta$

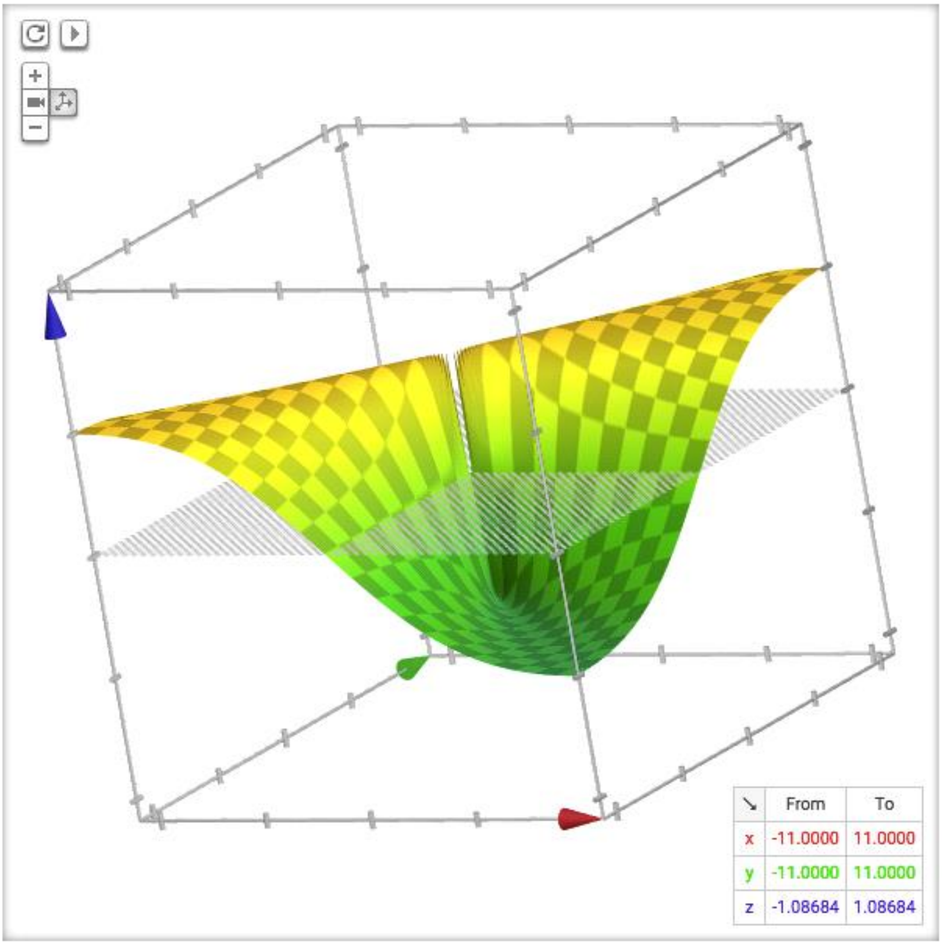


- When $\theta = \frac{\pi}{2} \Rightarrow f = 0$, when $\theta = \frac{\pi}{4} \Rightarrow f = \frac{1}{2} \cdots$
- Therefore we get the counter plot near origin



- And the graph near 0





Derivative

- Directional Derivative
  - $D_h f(x) = \nabla_h f(x) = f'(x;\vec{h}) = df_x \cdot h$
  - $= \lim_{t \to 0} \dfrac{f(x + t\vec{h}) - f(x)}{t}$
  - $= \left[\dfrac{d}{dt} f(x + t\vec{h})\right]_{t=0}$
- Example
  - $f: \mathbb{R}^n \to \mathbb{R}$

    - $f(x) = \|x\|^2$
    - $f'(x; \vec{h})$
    - $= \left[\frac{d}{dt} f(x + th)\right]_{t=0}$
    - $= \left[\frac{d}{dt} \|x + th\|^2\right]_{t=0}$
    - $= \left[\frac{d}{dt} (h^2 t^2 + (2h \cdot x)t + x^2)\right]_{t=0}$
    - $= [2h^2 t + 2h \cdot x]_{t=0}$
    - $= 2x \cdot h$
- Partial Derivative
- Total Derivative

11/29

Wednesday, November 29, 2017

Question 1 (from Monday)

- Given
  - Let V be a vector space and let $T: V \to V$ be a linear map
  - Suppose $x, y \in V$ are eigenvectors of T with eigenvalues λ and μ.
- Prove
  - If $ax + by$ $(a \neq 0, b \neq 0)$ is an eigenvector of T, then $\lambda = \mu$
- Proof
  - $Tx = \lambda x, \qquad Ty = \mu y$
  - $\Rightarrow T(ax + by) = a\lambda x + b\mu y$
  - Denote the eigenvalue for $ax + by$ to be k
  - $\Rightarrow T(ax + by) = k(ax + by)$
  - $\Rightarrow a\lambda x + b\mu y = akx + bky$
  - $\Rightarrow a(\lambda - k)x - b(\mu - k)y = 0$
  - If x, y are linearly independet
    - $a(\lambda - k) = b(\mu - k) = 0$
    - Because $a \neq 0, b \neq 0$
    - $\Rightarrow \lambda = \mu = k$
  - If x, y are linearly dependet
    - $x = cy$ for some c
    - $Tx = cTy = c\mu y = \mu(cy) = \mu x$
    - $\Rightarrow \lambda = \mu$

Question 2

- Given
  - Let A be a real $n \times n$ matrix such that $A^2 = -I$
- Note
  - $\begin{bmatrix} 0 & a \\ -1/a & 0 \end{bmatrix}^2 = -I, \qquad (a \neq 0)$
- Proof: A is invertibe
  - $A(-A) = -A^2 = -(-I) = I$
  - $\Rightarrow A^{-1} = -A$
  - $\Rightarrow A$ is invertibe
- Proof: n is even
  - Suppose n is odd
  - $\det A^2 = (\det A)^2 \geq 0$
  - $\det(-I) = -1 < 0$

- Which makes a contradiction
- Therefore n is even

- Proof: A has no real eigenvalues
  - Suppose $\exists \lambda \in \mathbb{R}, x \in \mathbb{R}^n, \text{s.t.}\ Ax = \lambda x$
  - $A^2 x = -Ix = -x = \lambda^2 x$
  - So $\lambda^2 = -1 \Rightarrow \lambda = \pm i$
  - Which makes a contradiction
  - Therefore A has no real eigenvalues
- Proof: $\det A = 1$ (when $n = 2$)
  - $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
  - $A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & d^2 + bc \end{bmatrix}$
  - $\begin{cases} a^2 + bc = d^2 + bc = -1 \\ ab + bd = ac + cd = 0 \end{cases}$
  - $\Rightarrow ad - bc = 1$
- Proof: $\det A = 1$ (general case)
  - $(\det A)^2 = \det A^2 = \det(-I) = (-1)^n = 1$
  - $\Rightarrow \det A = \pm 1$
  - $Ax = \lambda x \Rightarrow \overline{Ax} = \overline{\lambda x} \Rightarrow A\bar{x} = \bar{\lambda}\bar{x}$
  - Therefore the eigenvalues come in complex conjugate pairs
  - $\det A = (\lambda_1 \overline{\lambda_1})(\lambda_2 \overline{\lambda_2}) \cdots (\lambda_k \overline{\lambda_k}) \geq 0$
  - Therefore $\det A = 1$

Question 3

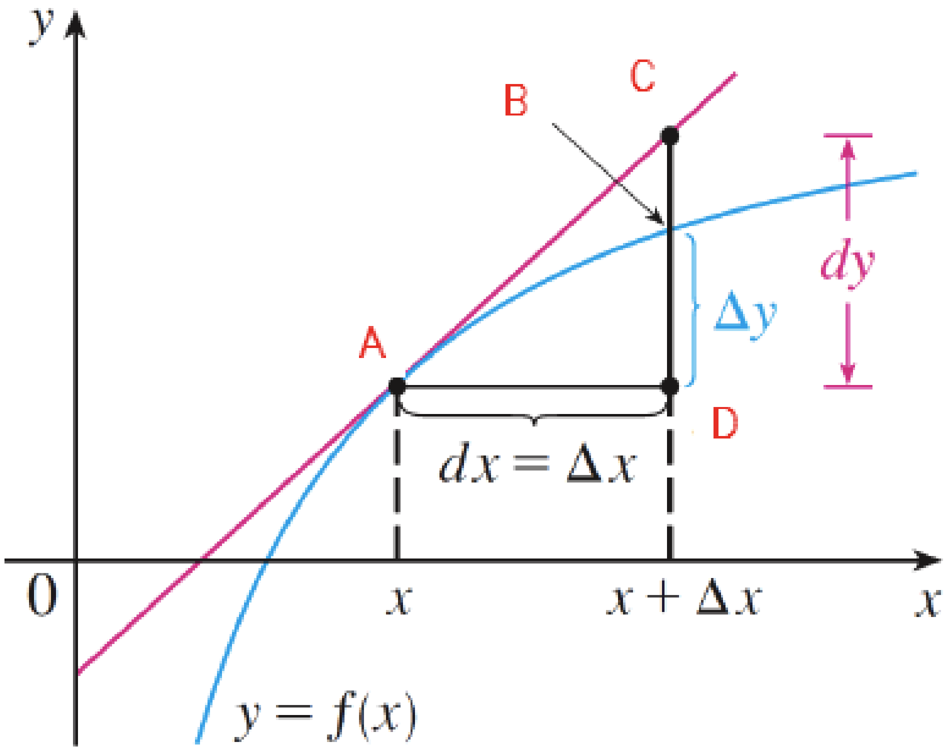
- Given
  - Let $T: V \to V$ be a finite-dimensional real linear transformation
  - T has no real eigenvalues
- Proof: n is even
  - Suppose n is odd
  - $f(\lambda) = -\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$
  - As $\lambda \to \infty, f(\lambda) \Rightarrow -\infty$
  - As $\lambda \to -\infty, f(\lambda) \Rightarrow \infty$
  - By the Intermediate Value Theorem
  - $f(\lambda)$ must have a real root
  - Which makes a contradiction
  - Therefore n is even
- Proof: $n = \dim V$

11/30

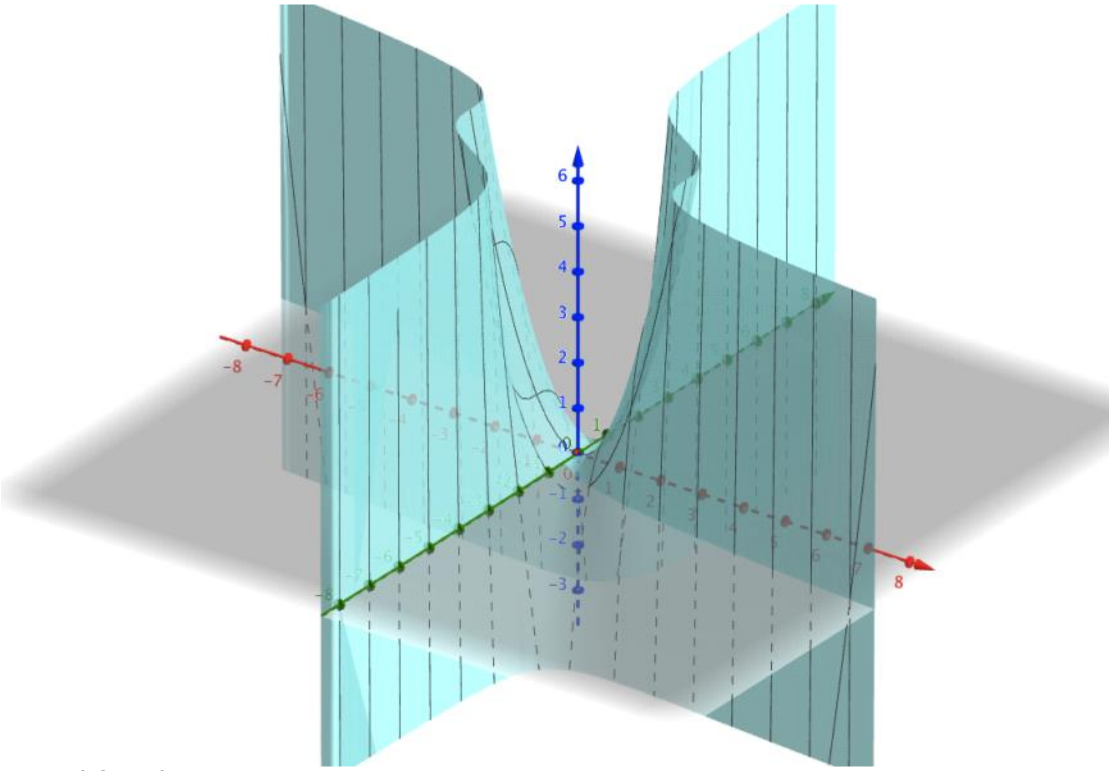
Thursday, November 30, 2017

Partial Derivative

- Infinitesimal Interpretation of Derivative

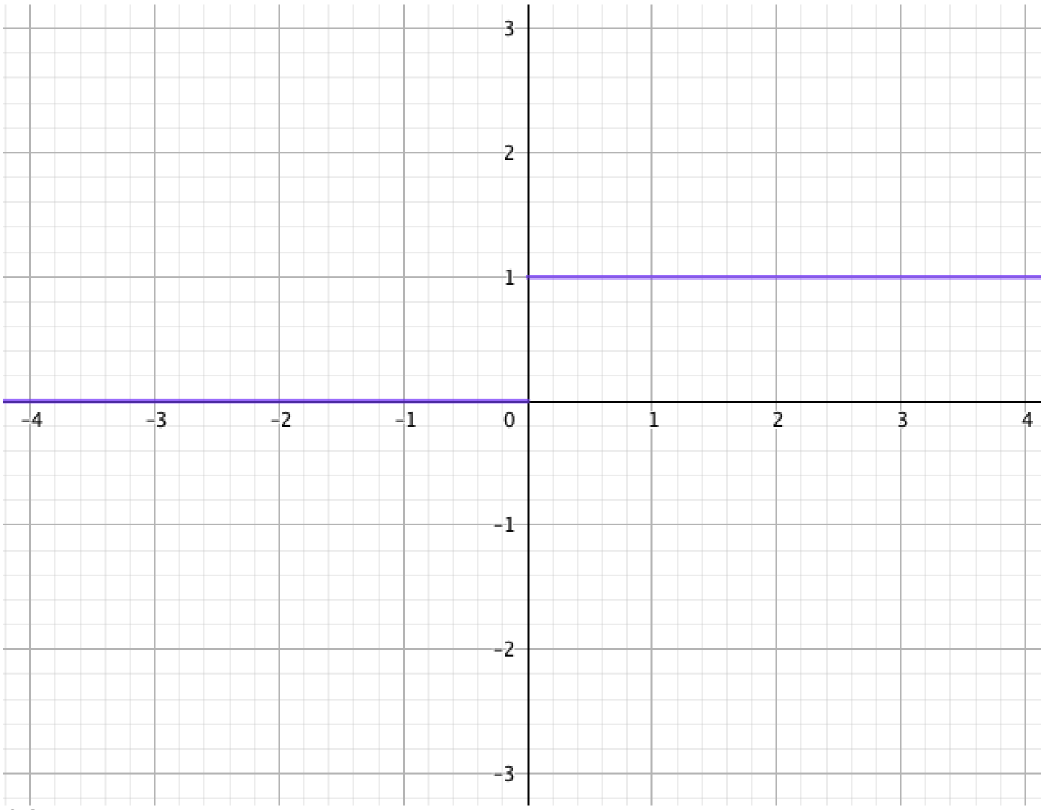


- Definition
    - $\frac{\partial f}{\partial x_k}(x_1, \dots, x_n) = \lim_{h \to 0} \frac{\overbrace{f(x_1, \dots, x_k + h, \dots, x_n)}^{\text{only } x_k \text{ changes}} - f(x_1, \dots, x_n)}{h}$
    - $=$ The derivative of $f(x_1, \dots x_n)$ with respect to x_k, with all other variables fixed
- Other Notations
    - $\frac{\partial f}{\partial x_k}(x_1, \dots, x_n) = f_{x_k} = f'(x; e_k)$
- Example
    - $f(x, y, z) = x^2 + xy^3$



- $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + xy^3) = 2x + y^3$
- $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + xy^3) = 3xy^2$
- $\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x^2 + xy^3) = 0$ (Because $x^2 + xy^3$ does not depend on z)

- Second Derivative
  - $f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x + y^3) = 2$
  - $f_{xy} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x + y^3) = 3y^2$
  - $f_{yx} = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3xy^2) = 3y^2$
  - $f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3xy^2) = 6xy$
- Clairaut's Theorem
  - If $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x\,\partial y}$ exists and $\frac{\partial^2 f}{\partial x\,\partial y}$ is continuous at $(a, b) \in \mathbb{R}^2$
  - Then $\frac{\partial^2 f}{\partial y\,\partial x}$ also exists and $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}$
- Example of $f_{xy} \neq f_{yx}$
  - $f(x, y) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$
  - $\frac{\partial f}{\partial x} = \begin{cases} 0 & x \neq 0 \\ \text{Does Not Exist} & x = 0 \end{cases}$
  - see the graph below (horizontal axis: x, vertical axis: $f(x, y)$)



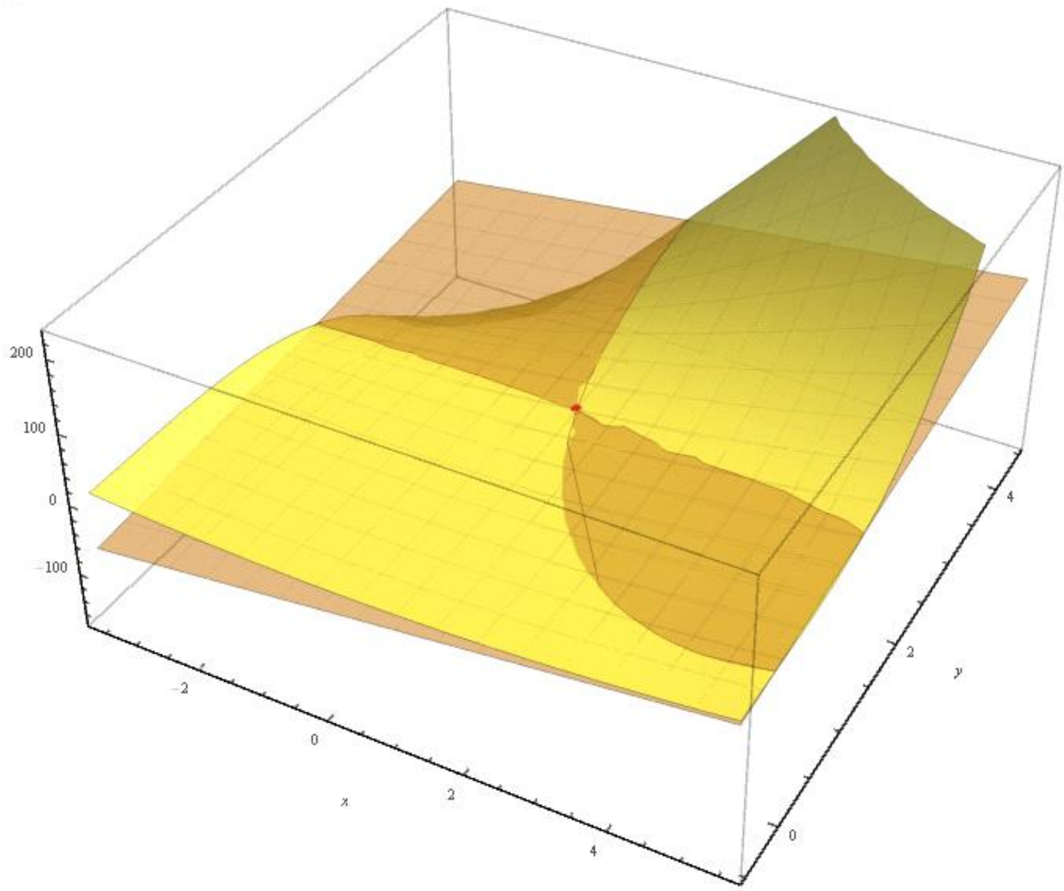
- $\frac{\partial f}{\partial y} = 0$ for all (x, y)
- $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(0) = 0$
- $\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \begin{cases} 0 & x \neq 0 \\ \text{Does Not Exist} & x = 0 \end{cases}$
- Therefore $\frac{\partial^2 f}{\partial x\,\partial y} \neq \frac{\partial^2 f}{\partial y\,\partial x}$

Total Derivative & Linear Approximation Formula

- Illumination
  - $f(x + \Delta x, y + \Delta y) - f(x, y)$
  - $= f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) + f(x + \Delta x, y) - f(x, y)$
  - $= [f(x + \Delta x, y) - f(x, y)] + [f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)]$
  - $= \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \times \Delta x + \frac{f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)}{\Delta y} \times \Delta y$
  - $\approx \frac{\partial f}{\partial x} \times \Delta x + \frac{\partial f}{\partial y} \times \Delta y$
- Theorem
  - If f_x and f_y are continuous, then there exist functions ε_x and ε_y
  - $f(x + \Delta x, y + \Delta y) = f(x, y) + \frac{\partial f}{\partial x}(x, y)\Delta x + \frac{\partial f}{\partial y}(x, y)\Delta y + \varepsilon_x \Delta x + \varepsilon_y \Delta y$
  - Where $\varepsilon_x, \varepsilon_y \to 0$ as $\Delta x, \Delta y \to 0$
  - Note
    - $\frac{f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)}{\Delta y} = \frac{\partial f}{\partial y}(x, y) + \varepsilon_y$

- $\dfrac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \dfrac{\partial f}{\partial x}(x,y) + \varepsilon_x$

- Linear Approximation
  - $f(x_1 + \Delta x_1, \dots, x_n + \Delta x_n)$
  - $= f(x_1, \dots, x_n) + f_{x_1}(x_1, \dots, x_n)\Delta x_1 + \dots + f_{x_n}(x_1, \dots, x_n)\Delta x_n + \varepsilon_1 \Delta x_1 + \dots + \varepsilon_n \Delta x_n$
  - Where $\varepsilon_k \to 0$ as $\Delta x_1, \dots, \Delta x_n \to 0$
- Linear Approximation (Vector Notation)
  - $x = (x_1, \dots, x_n) \in \mathbb{R}^n$
  - $\Delta x = (\Delta x_1, \dots, \Delta x_n) \in \mathbb{R}^n$
  - $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{R}^n$
  - $f(x + \Delta x) = f(x) + \vec{\nabla} f(x) \cdot \Delta x + \varepsilon \cdot \Delta x$
  - Where
    - $\vec{\nabla} f(x) = \left(\dfrac{\partial f}{\partial x_1}(x), \dots, \dfrac{\partial f}{\partial x_n}(x) \right)$ is called the gradient of f
    - $\vec{\nabla} f(x) \cdot \Delta x = f_{x_1}(x_1, \dots, x_n)\Delta x_1 + \dots + f_{x_n}(x_1, \dots, x_n)\Delta x_n$
    - $\varepsilon \cdot \Delta x = \varepsilon_1 \Delta x_1 + \dots + \varepsilon_n \Delta x_n$
- Example
  - $f(x,y) = x^2 + xy^3$
  - Find the linear approximation at $(x,y) = (1,2)$
  - Calculate $f(1,2), f_x(1,2), f_y(1,2)$
    - $f(1,2) = 1^2 + 1 \cdot 2^3 = 9$
    - $f_x(1,2) = [2x + y^3]_{\substack{x=1\\y=2}} = 2 + 2^3 = 10$
    - $f_y(1,2) = [3xy^2]_{\substack{x=1\\y=2}} = 3 \cdot 1 \cdot 2^2 = 12$
    - $\vec{\nabla} f(1,2) = \begin{bmatrix} 10 \\ 12 \end{bmatrix}$
  - $f(1 + \Delta x, 2 + \Delta y)$
    - $= f(1,2) + f_x(1,2)\Delta x + f_y(1,2)\Delta y + \varepsilon_x \Delta x + \varepsilon_y \Delta y$
    - $= \underbrace{9 + 10\Delta x + 12\Delta y}_{\text{approximation}} + \underbrace{\varepsilon_x \Delta x + \varepsilon_y \Delta y}_{\text{error}}$
  - $f(1.01, 1.99) = f(1 + 0.01, 2 - 0.01) \approx 9 + 10 \cdot 0.01 - 12 \cdot 0.01 = 8.89$
  - Tangent plane at $(1,2)$


200
100
0
−100
−2
0
2
4
x
0
2
4
y